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On Barenblatt's Model of Spontaneous Countercurrent Imbibition*

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Abstract. Water imbibition is a critical mechanism of secondary oil recovery from fractured reservoirs. Spontaneous imbibition also plays a significant role in storage of liquid waste by controlling the extent of rock invasion. In the present paper, we extend a model of countercurrent imbibition based on Barenblatt's theory of non-equilibrium two-phase flow by allowing the model's relaxation time to be a function of the wetting fluid saturation. We obtain two asymptotic self-similar solutions, valid at early and late times, respectively. At a very early stage, the time scale characterizing the cumulative volume of imbibed (and expelled) fluid is a power function with exponent between 1.5 and 1. At a later stage, the time scaling for this volume approaches asymptotically classical square root of time, whereas the saturation profile asymptotically converges to Ryzhik's self-similar solution. Our conclusions are verified against experiments. By fitting the laboratory data, we estimate the characteristic relaxation times for different pairs of liquids.

Key words: non-equilibrium two-phase flow, countercurrent imbibition, asymptotic solution.

1. Introduction

The model of non-equilibrium imbibition considered in the present work was developed by Barenblatt and his co-authors (Barenblatt, 1971; Barenblatt and Vinnichenko, 1980; Barenblatt and Gilman, 1987). The basic concepts of two-phase immiscible flow in this model are the same as in classical approach: the fluids flow through separate systems of flow paths (Muskat and Meres, 1936; Leverett, 1939) and the difference between the individual fluid pressures at a given location is characterized by the capillary pressure (Leverett, 1941). The latter is determined by the interaction between the fluids and the rock, and depends on the pore space geometry, physical properties of the fluids and the rock, and the fluid saturations. In classical approach, it is assumed that the phase permeabilities, as well as the capillary pressure, are functions of fluid saturations only. The relative permeability functions are usually obtained in steady-state flow experiments, and the capillary pressure curves are measured statically. Therefore, the classical model assumes an instantaneous redistribution of the fluids in the pore space with changing saturation.

The distinctive feature of Barenblatt's model is that the redistribution of the fluids in the pore space with changing saturations is not instantaneous, but takes a certain time. Therefore, in transient processes, where the redistribution time is comparable with the characteristic transition time, the relative permeabilities and capillary pressure cannot be universal functions of instantaneous fluid saturations only, but are *process-dependent* quantities: at a given saturation, their values are different from the relative permeabilities and capillary pressure measured in steady-state flow experiments.

Using the properties of relative permeability and capillary pressure functions, the concept of effective saturation was introduced in Barenblatt (1971), Barenblatt and Vinnichenko (1980) and Barenblatt and Gilman (1987), so that the relative permeabilities and capillary pressure are evaluated using the same curves as in the classical model, but at a certain *effective saturation* rather than at the actual one. It was assumed that the effective and actual saturations are related through the rate of change of the actual saturation. To characterize this relationship, a kinetic equation was derived and added to the model. Dimensional analysis and linearization of this equation leads to a simplified relationship, where the difference between the effective and actual saturations is equal to the product of the rate of growth of the actual saturation and the characteristic relaxation time, see Barenblatt (1971) and Barenblatt and Vinnichenko (1980). In the present paper, we use this linearized relationship.

In Barenblatt (1971), Barenblatt and Vinnichenko (1980), Barenblatt and Gilman (1987), Barenblatt *et al.* (1997b) and Natalini and Tesei (1999), non-equilibrium two-phase flow was studied under the assumption that the redistribution time does not depend on saturation. In particular, Barenblatt and Gilman (1987) investigated a mathematical model of non-equilibrium countercurrent imbibition with constant relaxation time. An asymptotic scaling of oil recovery at early times was obtained. This scaling is different from the classical square root of time behavior reported by Rapoport (1955).

In the present work, we assume that the relaxation time is *not* a constant, but rather a function of instantaneous saturation. More specifically, we assume that the relaxation time vanishes and blows up to infinity at irreducible water and residual oil saturations, respectively. Other than near the end-point saturations, the relaxation time is close to a constant. This assumption is partially justified by a heuristic argument in Barenblatt *et al.* (1990), where the relaxation time is related to the inverse of the capillary pressure derivative.

Here we assume zero initial water saturation. In such a case, the water saturation profile is *monotone* and propagates with a finite speed. We deduce that the asymptotic scalings of the actual and effective saturation profiles are self-similar near the tip of the curve.

We obtain two asymptotic solutions, which are valid, respectively, at early and late times. At early time, we neglect the actual saturation in comparison with the effective saturation in the imbibed zone. At large time, we drop the rapidly

decreasing terms. As a result, we obtain two approximate self-similar solutions. Using experimental data provided to us by A. R. Kovscek (Zhou *et al.*, 2001), we demonstrate that the measured cumulative non-wetting fluid recoveries obtained for different pairs of wetting–non-wetting fluids with different viscosities collapse to a single curve derived from our solution.

The paper is organized as follows. In Section 2, we formulate the initial and boundary-value problem. In Section 3, we analyze the inlet boundary condition, and in Section 4 we carry out the scaling of the propagating fluid saturation profile near its front end. In Section 5, we obtain a self-similar asymptotic solution valid at early times. In Section 6, we obtain an asymptotic solution for large times. We verify this solution by matching the oil recovery measurements in countercurrent imbibition of diatomite cores reported in Zhou *et al.* (2001). By matching the data, we also estimate the characteristic relaxation times for these experiments. In the concluding section, we outline an application of a boundary-layer method to the problem of countercurrent imbibition considered in this paper.

Over the past 50 years, numerous attempts to develop an appropriate model of spontaneous imbibition have been undertaken. A survey of major contributions in this area has been presented elsewhere (Barenblatt *et al.*, 2003).

2. Flow Equations

Assume that two immiscible fluids occupy the entire pore space, that is, the sum of their saturations equals 1. One fluid is wetting (water) and the other one is non-wetting (oil or NAPL). We neglect gravity as well as the compressibility of both fluids and of the rock. We also assume that the processes under consideration are isothermal.

Consider a horizontal half-infinite block of porous medium initially filled with oil. In other words, we consider a situation where the boundary conditions on the far end of the block play no significant role and can be replaced by a zero Dirichlet condition at infinity. Initially, the surface of the block is contacted by water, which starts imbibing spontaneously into the block. The advancing water displaces from the pore space an equal volume of oil, which flows back to the surface of the block and escapes through the inlet. Therefore, capillarity causes equal and opposite flows of the fluids, and this process is called *countercurrent* imbibition.

Denote by S_{w}, P_{w} and S_{n}, P_{n} the saturations and the pressures of the wetting and non-wetting fluid, respectively. Clearly,

$$S_{\mathrm{w}} + S_{\mathrm{n}} \equiv 1 \tag{2.1}$$

In what follows, we will omit the subscript w by setting $S = S_{\mathrm{w}}$. The wetting fluid saturation S can vary only between two end-point values. For simplicity, we assume that for both the actual and effective saturations the end-point values are 0 and 1. This assumption is not restrictive because it can be always satisfied after a renormalization.

At local equilibrium, the non-wetting fluid pressure is higher than that of the wetting fluid and the capillary pressure $P_c = P_n - P_w$ is positive. In the classical two-phase flow model, it is assumed that P_c is a function of S:

$$P_n - P_w = P_c(S). \tag{2.2}$$

The capillary pressure, $P_c(S)$, decreases monotonically to zero as S approaches complete saturation, and increases indefinitely as S tends to irreducible saturation. Leverett *et al.* (1942) represented $P_c(S)$ in the form

$$P_c(S) = \sigma\sqrt{\frac{\phi}{k}}J(S) \tag{2.3}$$

where σ is the oil–water interfacial surface tension coefficient, and ϕ and k are, respectively, the porosity and the permeability of the rock. The dimensionless Leverett's J-function is often assumed to be universal, although there is little evidence that it is the same for different types of rocks and different pairs of fluids.

The horizontal flow of each fluid is characterized by the superficial velocity u. The latter depends on the respective pressure gradient according to Darcy's law:

$$u_i = -\frac{kk_{ri}(S)}{\mu_i}\nabla P_i \tag{2.4}$$

Here $i = w, n$ and k_{ri} and μ_i are, respectively, the relative permeability and viscosity of fluid i. In immiscible flow, each fluid flows along its own system of flow paths. The configurations of these paths are different at different saturations. Therefore, in the classical model, both relative permeability coefficients k_{ri} are functions of S.

Equation (2.4) holds true for immiscible flow at steady-state conditions, where the wetting fluid flows in narrower pores and corners. Such a configuration corresponds to the minimal possible permeability at a given saturation. If the fluid saturations change, the fluids rearrange themselves accordingly in the pore space. During the transition, the wetting fluid relative permeability is different from the one at steady-state conditions at the same saturation. Therefore, the non-equilibrium relative permeability of the wetting fluid is higher than the steady-state one. Similarly, during the redistribution of the flow paths both the relative permeability of the non-wetting fluid and the capillary pressure are lower than in steady-state flow. Strictly speaking, this reasoning implies that the relative permeability and capillary pressure functions obtained in steady-state flow experiments *cannot* be used in transient processes whose characteristic transition times are comparable with characteristic fluid redistribution times. However, due to the monotonicity of these functions, Figure 1, they still can be used to characterize transient flow. To do so, the relative permeabilities and the capillary pressure must to be evaluated not at actual instantaneous values of saturation S, but at some effective saturations $\eta \geqslant S$, see Figure 1. In general, the effective saturation η can be different for each function k_{ri} and J. Following Barenblatt (1971), Barenblatt and Vinnichenko (1980) and

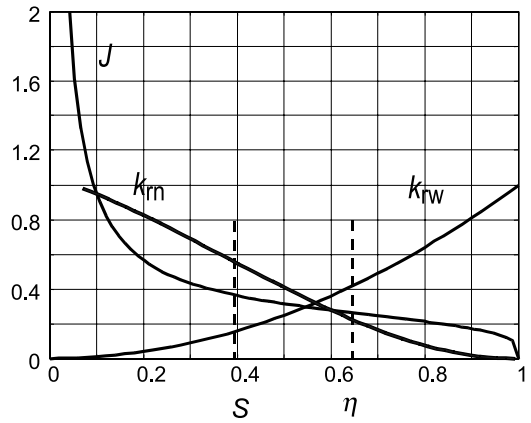

Figure 1. Effective saturation is higher than the actual one due to the monotonicity of the curves.

Barenblatt and Gilman (1987), we assume that the effective saturation is the same for all three functions.

Thus, relationships (2.2) and (2.4) transform into the following equations:

$$P_{\mathrm{n}} - P_{\mathrm{w}} = P_{\mathrm{c}}(\eta) \tag{2.5}$$

and

$$u_i = -\frac{k k_{\mathrm{r}i}(\eta)}{\mu_i} \nabla P_i \tag{2.6}$$

where $i = \mathrm{w}, \mathrm{n}$ as above.

Due to the non-equilibrium effects, the relationship between η and S must be a process-dependent function. We adopt the hypothesis (Barenblatt, 1971; Barenblatt and Vinnichenko, 1980) that there is a relationship between the local effective saturation η and the actual saturation S and its rate of change $\partial S/\partial t$. Dimensional analysis suggests that such a relationship must include a characteristic redistribution time. Further, linearization of this relationship yields

$$\tau \frac{\partial S}{\partial t} = \eta - S \tag{2.7}$$

Here τ is a coefficient having the dimension of time. If effective saturation η were fixed and τ were constant, then the difference $\eta - S$ would decay exponentially as $\exp(-t/\tau)$. Therefore, τ is a characteristic relaxation time needed for the rearrangement of the menisci and flow paths to a new steady-state configuration. Note that in spontaneous imbibition such a steady state is *never* attained.

Generally speaking, the redistribution time may depend on the properties of the rock and fluids. In the case of spontaneous imbibition, where the driving force is the capillary pressure, we assume $\tau = \tau(S)$. At low saturations, a small volume of the wetting fluid flows through the narrowest flow paths along the pore corners

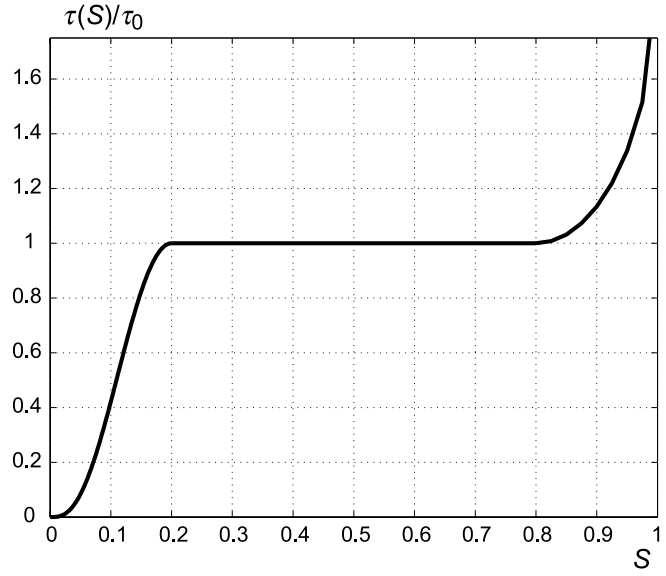

Figure 2. Qualitative plot of $\tau(S)$ normalized by constant τ_0.

and the capillary pressure is high. Therefore, the time needed to reconfigure water distribution is small and the relaxation time goes to zero as $S \to 0^+$. At a mature stage of imbibition, as the oil saturation approaches its residual value, the capillary forces weaken and the connectivity of the oil clusters grows sparse. Hence, the redistribution time blows up to infinity. Across most of the interval between the end-point saturations, the variation of relaxation time is relatively small. Let us denote the relaxation time on the major part of the saturation interval by τ_0, see Figure 2.

We assume that both fluids are incompressible and there is no internal source or sink of either fluid. Thus, the mass balance equations are:

$$\phi \frac{\partial S}{\partial t} + \nabla \cdot u_{\mathrm{w}} = 0 \tag{2.8}$$

$$\phi \frac{\partial (1 - S)}{\partial t} + \nabla \cdot u_{\mathrm{n}} = 0 \tag{2.9}$$

Equations (2.5)–(2.9) are coupled and involve six unknown functions: the effective and instantaneous water saturations, the water and oil pressures, and the water and oil superficial velocities.

Now, let us exclude the pressures in order to obtain a unique equation involving the saturations and volumetric fluxes only. We assume that initially the porous block is fully saturated with the non-wetting fluid and is later imbibed by the wetting fluid through an inlet face. The porous block is large enough, therefore, far from the lateral boundaries and the edges of the inlet face, the flow is only

in the direction orthogonal to the inlet face. Denote by x the coordinate in this direction. After replacing ∇ with $\partial/\partial x$, let us sum up equation (2.6) for $i = \mathrm{w}, \mathrm{n}$:

$$u_{\mathrm{w}} + u_{\mathrm{n}} = -\lambda_{\mathrm{n}} \frac{\partial P_{\mathrm{n}}}{\partial x} - \lambda_{\mathrm{w}} \frac{\partial P_{\mathrm{w}}}{\partial x} \tag{2.10}$$

where

$$\lambda_{\mathrm{w}}(\eta) = \frac{k k_{\mathrm{rw}}(\eta)}{\mu_{\mathrm{w}}} \quad \text{and} \quad \lambda_{\mathrm{n}}(\eta) = \frac{k k_{\mathrm{rn}}(\eta)}{\mu_{\mathrm{n}}} \tag{2.11}$$

Let us further transform the right-hand side of Equation (2.10) by introducing an 'average' pressure P, defined in such a way that

$$\frac{\lambda_{\mathrm{n}}(\eta)}{\lambda_{\mathrm{n}}(\eta) + \lambda_{\mathrm{w}}(\eta)} \frac{\partial P_{\mathrm{n}}}{\partial x} + \frac{\lambda_{\mathrm{w}}(\eta)}{\lambda_{\mathrm{n}}(\eta) + \lambda_{\mathrm{w}}(\eta)} \frac{\partial P_{\mathrm{w}}}{\partial x} = \frac{\partial P}{\partial x} \tag{2.12}$$

Using equality (2.2), one infers that

$$\frac{\partial P}{\partial x} = \frac{\partial P_{\mathrm{w}}}{\partial x} + \frac{\lambda_{\mathrm{n}}(\eta)}{\lambda_{\mathrm{n}}(\eta) + \lambda_{\mathrm{w}}(\eta)} \frac{\partial P_{\mathrm{c}}}{\partial x} = \frac{\partial P_{\mathrm{n}}}{\partial x} - \frac{\lambda_{\mathrm{w}}(\eta)}{\lambda_{\mathrm{n}}(\eta) + \lambda_{\mathrm{w}}(\eta)} \frac{\partial P_{\mathrm{c}}}{\partial x} \tag{2.13}$$

Therefore, up to a constant,

$$P = P_{\mathrm{w}} - \int_{\eta}^{1} P_{\mathrm{c}}'(\zeta) \left(\frac{\lambda_{\mathrm{n}}(\zeta)}{\lambda_{\mathrm{n}}(\zeta) + \lambda_{\mathrm{w}}(\zeta)} \right) \mathrm{d}\zeta \tag{2.14}$$

Equivalently,

$$P = P_{\mathrm{n}} + \int_{\eta}^{1} P_{\mathrm{c}}'(\zeta) \left(\frac{\lambda_{\mathrm{w}}(\zeta)}{\lambda_0(\zeta) + \lambda_{\mathrm{w}}(\zeta)} \right) \mathrm{d}\zeta \tag{2.15}$$

Now, combining Equations (2.14) and (2.15) with (2.6) for $i = \mathrm{w}, \mathrm{n}$, and using the identities

$$\begin{aligned} &\lambda_{\mathrm{w}}(\eta) \nabla \int_{\eta}^{1} P_{\mathrm{c}}'(\zeta) \left(\frac{\lambda_{\mathrm{n}}(\zeta)}{\lambda_{\mathrm{n}}(\zeta) + \lambda_{\mathrm{w}}(\zeta)} \right) \mathrm{d}\zeta \\ &\quad = \nabla \int_{\eta}^{1} P_{\mathrm{c}}'(\zeta) \left(\frac{\lambda_{\mathrm{w}}(\zeta) \lambda_{\mathrm{n}}(\zeta)}{\lambda_{\mathrm{n}}(\zeta) + \lambda_{\mathrm{w}}(\zeta)} \right) \mathrm{d}\zeta \\ &\quad = \lambda_{\mathrm{n}}(\eta) \nabla \int_{\eta}^{1} P_{\mathrm{c}}'(\zeta) \left(\frac{\lambda_{\mathrm{w}}(\zeta)}{\lambda_{\mathrm{n}}(\zeta) + \lambda_{\mathrm{w}}(\zeta)} \right) \mathrm{d}\zeta \end{aligned} \tag{2.16}$$

one infers that

$$u_{\mathrm{w}} = -\lambda_{\mathrm{w}}(\eta) \frac{\partial P}{\partial x} - \phi a^2 \frac{\partial \Phi(\eta)}{\partial x} \tag{2.17}$$

$$u_{\mathrm{n}} = -\lambda_{\mathrm{n}}(\eta) \frac{\partial P}{\partial x} + \phi a^2 \frac{\partial \Phi(\eta)}{\partial x} \tag{2.18}$$

Here, by virtue of Equation (2.3),

$$\Phi(\eta) = \Phi_0 + \int_\eta^1 J'(\zeta) k_{\mathrm{rw}}(\zeta) \frac{k_{\mathrm{rn}}(\zeta)}{k_{\mathrm{rw}}(\zeta)\frac{\mu_{\mathrm{n}}}{\mu_{\mathrm{w}}} + k_{\mathrm{rn}}(\zeta)}\, \mathrm{d}\zeta \tag{2.19}$$

is a dimensionless function and

$$a^2 = \frac{\sigma}{\mu_{\mathrm{w}}}\sqrt{\frac{k}{\phi}} \tag{2.20}$$

We select the constant Φ_0 in such a way that $\Phi(0) = 0$. A simple calculation yields

$$\Phi(\eta) = -\int_0^\eta J'(\zeta) k_{\mathrm{rw}}(\zeta) \frac{k_{\mathrm{rn}}(\zeta)}{k_{\mathrm{rw}}(\zeta)(\mu_{\mathrm{n}}/\mu_{\mathrm{w}}) + k_{\mathrm{rn}}(\zeta)}\, \mathrm{d}\zeta \tag{2.21}$$

From Equation (2.21), the function Φ increases between $\eta = 0$ and $\eta = 1$. Since the product $k_{\mathrm{rw}}(\eta)k_{\mathrm{rn}}(\eta)$ vanishes only at $\eta = 0$ and $\eta = 1$, and the derivative of Leverett's J-function is negative, function $\Phi(\eta)$ increases monotonically. It attains the minimum at $\eta = 0$ and the maximum at $\eta = 1$. At both end points the derivative of function Φ vanishes. This behavior is implied by the finite speed of the wetting fluid propagation.

Since the fluids flow only in the x-direction, the total volumetric flux is identically equal to zero:[1]

$$u_{\mathrm{w}} + u_{\mathrm{n}} \equiv 0 \tag{2.22}$$

Hence, Equations (2.10) and (2.12) imply that $\partial P/\partial x \equiv 0$ and Equations (2.17) and (2.18) reduce to

$$u_{\mathrm{w}} = -\phi a^2 \frac{\partial \Phi(\eta)}{\partial x} \tag{2.23}$$

$$u_{\mathrm{n}} = \phi a^2 \frac{\partial \Phi(\eta)}{\partial x} \tag{2.24}$$

Substitution of equality (2.23) into (2.8) in combination with (2.7) yields a system of two equations

$$\tau(S)\frac{\partial S}{\partial t} = \eta - S \tag{2.25}$$

$$\frac{\partial S}{\partial t} = a^2 \frac{\partial^2 \Phi(\eta)}{\partial x^2} \tag{2.26}$$

Using Equation (2.7), the last equation can be reformulated as

$$\frac{\eta - S}{\tau} = a^2 \frac{\partial^2 \Phi(\eta)}{\partial x^2} \tag{2.27}$$

[1]The authors are grateful to Prof V. Ryzhik for pointing out in his review that if the flow is not essentially one-dimensional, the total flux may be different from zero.

Either system of equations, (2.25) and (2.26), or (2.25) and (2.27), constitutes the mathematical model of non-equilibrium countercurrent imbibition.

Note that by virtue of Equation (2.25), at any $t > 0$ and an arbitrary location inside the block, effective and actual saturations are both positive or equal to zero simultaneously. Indeed, if at a certain time t_0 the effective saturation is positive on an interval where $S = 0$, then according to Equation (2.25) the derivative $\partial S/\partial t$ also must be positive on the same interval, and the actual saturation S must be greater than zero for all $t > t_0$. Hence, one obtains an infinite speed of the wetting fluid propagation, which is non-physical. Thus, if one of the functions is non-zero on an interval $(0, x(t))$ and is equal to zero elsewhere, then the other one is positive on the same interval and vanishes everywhere else.

For constant relaxation time, Barenblatt and Gilman (1987) proposed a transformation reducing this system of two partial differential equations to a single equation of third order. Such a transformation, in general, does not yield a single equation if $\tau = \tau(S)$.

The differential equations must be complemented by initial and boundary conditions. Initially, the block is saturated by the non-wetting fluid, therefore

$$S(x, 0) = 0, \quad x \geqslant 0 \tag{2.28}$$

If initial effective saturation at a certain distance from the inlet were positive, then by virtue of Equation (2.7) the actual saturation would instantaneously become positive at the same distance. This, in turn, implies an infinite speed of propagation, which is impossible. Hence,

$$\eta(x, 0) = 0, \quad x > 0 \tag{2.29}$$

At the inlet surface, the block contacts the wetting fluid from the very beginning of the process. Equation (2.26) implies that the effective saturation must be continuous with respect to x. Immediately outside the inlet face the actual saturation is identically equal to 1. Hence, the effective saturation at the inlet also equals 1:

$$\eta(0, t) = 1 \tag{2.30}$$

We consider the flow while the influence of all boundaries of the block except the inlet can be neglected. Therefore, one can formulate the boundary condition at infinity:

$$S(\infty, t) = \eta(\infty, t) = 0 \tag{2.31}$$

Equation (2.26) admits a solution that is non-zero only on a finite interval adjacent to the inlet, and the length of this interval grows with time. In order to be a solution to a differential equation (2.26) of second order, the composite function $\Phi(\eta)$ must be differentiable in x for all $t > 0$. Denote by $x(t)$ the coordinate of the tip of the saturation profile at time t. Then, at $x = x(t)$ one gets the following two boundary conditions:

$$S(x, t)|_{x=x(t)} = \eta(x, t)|_{x=x(t)} = 0 \tag{2.32}$$

and

$$\left.\frac{\partial \Phi(\eta(x,t))}{\partial x}\right|_{x=x(t)} = 0 \tag{2.33}$$

3. The Inlet Boundary Condition

In classical approach, the inlet boundary condition is usually formulated as constant wetting fluid saturation equal to 1 (Ryzhik, 1960). In particular, this implies that the non-wetting phase relative permeability vanishes at the inlet face of the block, which is non-physical.

In the model considered in this paper, the actual saturation S has a discontinuity at $x = 0$. It is identically equal to 1 outside the block and the interior limiting value is a function of time. Indeed, Equation (2.7) reduces to an ordinary differential equation

$$\tau(S(t))\frac{\mathrm{d}S(t)}{\mathrm{d}t} = 1 - S(t) \tag{3.1}$$

where $S(t, 0)$ is denoted by $S(t)$ for brevity. Initially, the wetting fluid saturation is zero. Therefore, one needs to solve a Cauchy problem with the initial condition

$$S(0) = 0 \tag{3.2}$$

It is easier to solve Equation (3.1) with respect to $t(S)$, that is, calculate the time when a particular saturation level at the inlet face is attained. Therefore, the Cauchy problem takes on the following form:

$$\frac{\mathrm{d}t(S)}{\mathrm{d}S} = \frac{\tau(S)}{1-S}, \quad t(0) = 0 \tag{3.3}$$

The solution is easily obtained by explicit integration

$$t(S) = \int_0^S \frac{\tau(\zeta)}{1-\zeta}\,\mathrm{d}\zeta \tag{3.4}$$

For small values of S, one has

$$t(S) \approx \int_0^S \tau(\zeta)\,\mathrm{d}\zeta \tag{3.5}$$

If the relaxation time is constant: $\tau(S) \equiv \tau_0$, then one deduces from Equation (3.4) that

$$S(t) = 1 - \mathrm{e}^{-t/\tau_0} \tag{3.6}$$

We have thus arrived at the boundary condition obtained by Barenblatt and Gilman (1987).

Bech *et al.* (1991) called a condition similar to (3.6) 'delayed imbibition' and used it for numerical simulations. In the simulations, though, the flow equations were based on the classical model.

4. Scaling of the Wetting Fluid Saturation Profile Near the Tip

In this section, we obtain the scaling of the wetting fluid saturation profile at the tip. As we have already mentioned in Section 1, at any given time $t > 0$ the solution is non-zero only on a finite interval $(0, x(t))$ and vanishes for all $x \geqslant x(t)$. Let us introduce new variables:

$$\vartheta = \frac{t}{\tau_0} \tag{4.1}$$

and

$$\xi = \frac{x}{x(t)} \tag{4.2}$$

Here τ_0 is the constant value of the relaxation time over the major part of the saturation interval, see Figure 2. As we have demonstrated above, $\tau(S)$ tends to zero as $S \to 0$. Assume, further, that near $S = 0$ the function $T(S) = \tau(S)/\tau_0$ can be expanded as

$$\frac{T(S)}{\tau_0} = T_* S^{n_\tau} + \cdots \tag{4.3}$$

where n_τ, T_* are positive numbers and the dots denote terms of higher order. Similarly, let us assume that the following approximation of Φ holds true at S close to zero:

$$\Phi(S) = \Phi_* S^{n_\Phi} + \cdots \tag{4.4}$$

As in Equation (4.3), Φ_* and n_Φ are positive numbers and the terms of higher order are denoted by the dots.

In what follows, we will use approximations like (4.3) and (4.4) many times. We will use the following rule for the notations. For an asymptotic expansion near the zero of a function of saturation, the power will be denoted by n with the respective subscript index, as in Equation (4.4). For an asymptotic power expansion of a function of dimensionless variable ξ introduced in Equation (4.2), we will use d with the respective subscript.

In variables (ϑ, ξ), the partial derivatives take the following form:

$$\tau_0 \frac{\partial}{\partial t} = \frac{\partial}{\partial \vartheta} - \frac{\xi}{X(\vartheta)} \frac{\mathrm{d}X(\vartheta)}{\mathrm{d}\vartheta} \frac{\partial}{\partial \xi} \tag{4.5}$$

and

$$\frac{\partial^2}{\partial x^2} = \frac{1}{X(\vartheta)^2} \frac{\partial^2}{\partial \xi^2} \tag{4.6}$$

where

$$X(\vartheta) = x(\tau_0 \vartheta) \tag{4.7}$$

Substitution of Equations (4.5) and (4.6) into Equations (2.25) and (2.26) yields

$$T(S)\left(\frac{\partial S(\vartheta,\xi)}{\partial \vartheta} - \frac{\xi}{X(\vartheta)}\frac{\mathrm{d}X(\vartheta)}{\mathrm{d}\vartheta}\frac{\partial S(\vartheta,\xi)}{\partial \xi}\right) = \eta - S \tag{4.8}$$

$$\frac{\partial S(\vartheta,\xi)}{\partial \vartheta} - \frac{\xi}{X(\vartheta)}\frac{\mathrm{d}X(\vartheta)}{\mathrm{d}\vartheta}\frac{\partial S(\vartheta,\xi)}{\partial \xi} = \frac{\tau_0 a^2}{X(\vartheta)^2}\frac{\partial^2}{\partial \xi^2}\Phi(\eta) \tag{4.9}$$

From (2.32), at $\xi = 1$ both $\eta(\vartheta,\xi)$ and $S(\vartheta,\xi)$, as functions of ϑ, are identically equal to zero. Let us investigate the behavior of these functions in a small left neighborhood of $\xi = 1$. More specifically, let us calculate the scalings of both actual and effective saturation profiles near the tip. For each ϑ, assume

$$S(\vartheta,\xi) = S_*(\vartheta)(1-\xi)^{d_S} + \cdots \tag{4.10}$$

and

$$\eta(\vartheta,\xi) = \eta_*(\vartheta)(1-\xi)^{d_\eta} + \cdots \tag{4.11}$$

where the exponents d_S and d_η, as well as functions $S_*(\vartheta)$ and $\eta_*(\vartheta)$, $\vartheta > 0$, are as yet unknown. Let us substitute Equations (4.10) and (4.11) along with (4.3) and (4.4) into Equations (4.8) and (4.9):

$$\begin{aligned} T_*(S_*(\vartheta)(1-\xi)^{d_S})^{n_\tau}&\left(\frac{\mathrm{d}S_*(\vartheta)}{\mathrm{d}\vartheta}(1-\xi)^{d_S} + \right.\\ &\left. + S_*(\vartheta)d_S(1-\xi)^{d_S-1}\frac{\xi}{X(\vartheta)}\frac{X(\vartheta)}{\mathrm{d}\vartheta}\right)\\ &= \eta_*(\vartheta)(1-\xi)^{d_\eta} - S_*(\vartheta)(1-\xi)^{d_S} + \cdots \end{aligned} \tag{4.12}$$

$$\begin{aligned} &\frac{\mathrm{d}S_*(\vartheta)}{\mathrm{d}\vartheta}(1-\xi)^{d_S} + S_*(\vartheta)d_S(1-\xi)^{d_S-1}\frac{\xi}{X(\vartheta)}\frac{\mathrm{d}X(\vartheta)}{\mathrm{d}\vartheta}\\ &\quad = \frac{\tau_0 a^2}{X(\vartheta)^2}\Phi_*\eta_*(\vartheta)^{n_\Phi}n_\Phi d_\eta(n_\Phi d_\eta - 1)(1-\xi)^{n_\Phi d_\eta - 2} + \cdots \end{aligned} \tag{4.13}$$

Equating the lowest order terms in Equations (4.12) and (4.13), one obtains

$$d_S = \frac{n_\Phi + 1}{n_\Phi(n_\tau + 1) - 1} \quad \text{and} \quad d_\eta = \frac{n_\tau + 2}{n_\Phi(n_\tau + 1) - 1} \tag{4.14}$$

In the original dimensional variables, near the tip of the saturation profile, one has

$$S(t,x) = S_*\left(\frac{t}{\tau_0}\right)\left(\frac{x(t) - x}{x(t)}\right)^{d_S} + \cdots \tag{4.15}$$

and

$$\eta(t,x) = \eta_* \left(\frac{t}{\tau_0}\right) \left(\frac{x(t)-x}{x(t)}\right)^{d_\eta} + \cdots \tag{4.16}$$

As usual, the dots denote terms of higher order. Physically, as the instantaneous wetting fluid saturation is delayed with respect to the effective saturation, one must have

$$d_S \geqslant d_\eta \tag{4.17}$$

and so, from Equation (4.14), $n_\Phi \geqslant n_\tau + 1$. Dividing Equations (4.12) and (4.13) by $(1-\xi)^{d_\eta}$ and $(1-\xi)^{d_S-1}$, respectively, and passing to the limit as $\xi \to 1-0$, one obtains

$$\frac{1}{X(\vartheta)}\frac{\mathrm{d}X(\vartheta)}{\mathrm{d}\vartheta} = \frac{\eta_*(\vartheta)}{T_* d_S S_*(\vartheta)^{1+n_\tau}} \tag{4.18}$$

$$\frac{\mathrm{d}X(\vartheta)^2}{\mathrm{d}\vartheta} = \tau_0 a^2 \Phi_* n_\Phi d_\eta (n_\Phi d_\eta - 1)\frac{\eta_*(\vartheta)^{n_\Phi}}{d_S S_*(\vartheta)} \tag{4.19}$$

Eliminating the derivative of $X(\vartheta)$, one gets

$$S_*(\vartheta)^{n_\tau} = \frac{\tau_0 a^2}{X(\vartheta)^2} T_* \Phi_* n_\Phi d_\eta (n_\Phi d_\eta - 1)\eta_*(\vartheta)^{n_\Phi - 1} \tag{4.20}$$

Thus, from the three functions $X(\vartheta)$, $S_*(\vartheta)$ and $\eta_*(\vartheta)$, each one determines the other two through the algebraic relationship (4.20), and any one of the differential equations (4.18) and (4.19).

5. Self-Similar Asymptotic Solution at Early Times

In this section, we obtain an approximate early-time self-similar solution to the system of equations (2.25) and (2.26) with the initial and boundary conditions (2.28)–(2.33).

Let us seek a solution in the following form:

$$S(t,x) = \vartheta^{\alpha_S} f_S(\xi) \tag{5.1}$$

$$\eta(t,x) = f_\eta(\xi) \tag{5.2}$$

where ϑ and ξ are the parameters defined in Equations (4.1) and (4.2) and α_S is an unknown parameter. Both f_S and f_η are unknown dimensionless functions of a scalar variable. Such form of solution is suggested by the incomplete similarity method (Barenblatt, 1996). We further assume that

$$x(t) = aB\left(\frac{t}{\tau_0}\right)^{\beta} \tag{5.3}$$

so that

$$\xi = \frac{x}{aB(t/\tau_0)^\beta} \tag{5.4}$$

and

$$X(\vartheta) = aB\vartheta^\beta \tag{5.5}$$

The coefficient B and exponent β are yet to be determined. Among all the unknown parameters only B is dimensional: $[B] = [t]^{1/2}$. Let us substitute Equations (5.1) and (5.2) into (2.25) and (2.26), and write the resulting equations in variables ϑ and ξ:

$$T(\vartheta^{\alpha_S} f_S(\xi)) \left(\alpha_S \vartheta^{\alpha_S - 1} f_S(\xi) - \beta \vartheta^{\alpha_S} f_S'(\xi) \frac{\xi}{\vartheta} \right) = f_\eta(\xi) - \vartheta^{\alpha_S} f_S(\xi) \tag{5.6}$$

$$\left(\alpha_S \vartheta^{\alpha_S - 1} f_S(\xi) - \beta \vartheta^{\alpha_S} f_S'(\xi) \frac{\xi}{\vartheta} \right) = \frac{\tau_0 a^2}{X(\vartheta)^2} \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \Phi(f_\eta(\xi)) \tag{5.7}$$

According to our assumptions, both functions f_S and f_η take positive values only for ξ varying between zero and 1, whereas ϑ is a small number. After substitution of expansion (4.3) into Equation (5.6), the comparison between the powers of ϑ on both sides of Equations (5.6) and (5.7) yields

$$\alpha_S = \frac{1}{1 + n_\tau} \tag{5.8}$$

and

$$\beta = \frac{1}{2} \frac{n_\tau}{1 + n_\tau} \tag{5.9}$$

In particular, for small times, the inlet condition (3.5) transforms into

$$S(t) = \left(\frac{t/\tau_0}{(n_\tau + 1)\, T_*} \right)^{1/(n_\tau + 1)} + \cdots \tag{5.10}$$

Due to relations (5.8) and (5.9), the similar powers of t in (5.6) and (5.7) cancel, and one obtains the following system of ordinary differential equations for the unknown functions f_S and f_η:

$$T_* f_S(\xi)^{n_\tau} \left[\alpha_S f_S(\xi) - \beta f_S'(\xi) \xi \right] = f_\eta(\xi) - \vartheta^{\alpha_S} f_S(\xi) \tag{5.11}$$

$$B^2 \left[\alpha_S f_S(\xi) - \beta f_S'(\xi) \xi \right] = \tau_0 \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \Phi(f_\eta(\xi)) \tag{5.12}$$

For small ϑ, the second term on the right-hand side of Equation (5.11) can be neglected in comparison with the other terms. Therefore, this equation can be approximated with its truncated form independent of ϑ:

$$T_* f_S(\xi)^{n_\tau} \left[\alpha_S f_S(\xi) - \beta f_S'(\xi)\xi\right] \approx f_\eta(\xi) \tag{5.13}$$

The boundary conditions (2.32) along with the definition of $x(t)$ and ξ imply

$$f_S(1) = f_\eta(1) = 0 \tag{5.14}$$

To satisfy the inlet boundary condition (2.29), put

$$f_\eta(0) = 1 \tag{5.15}$$

Then, by setting $\xi = 0$ in Equation (5.13), one infers that

$$f_S(0) = \left(\frac{n_\tau + 1}{T_*}\right)^{1/(n_\tau+1)} \tag{5.16}$$

It follows from (2.33) that:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\Phi(f_\eta(\xi))\bigg|_{\xi=1} = 0 \tag{5.17}$$

For ξ close to 1, we have obtained expansions (4.15) and (4.16). For small times, one can neglect the variation of the coefficients S_* and η_*. Thus, one obtains the following asymptotic expansions for the function f_S and f_η:

$$f_\eta(\xi) = \eta_*(1-\xi)^{d_\eta} + \cdots \tag{5.18}$$

$$f_S(\xi) = S_*(1-\xi)^{d_S} + \cdots \tag{5.19}$$

Here η_*, d_η, S_* and d_S are positive real numbers and the dots, as usual, denote higher order terms. The exponents d_η and d_S are obtained in Equations (4.14). Substitution of these results into the system of equations (5.12) and (5.13) yields

$$T_* S_*^{n_\tau+1}(1-\xi)^{(n_\tau+1)d_S-1}\left[\alpha_S(1-\xi) + \beta d_S\right] = \eta_*(1-\xi)^{d_\eta} \tag{5.20}$$

$$\begin{aligned} &B^2\left[\alpha_S(1-\xi) + \beta d_S\right] S_*(1-\xi)^{d_S-1} \\ &\quad = \tau_0 n_\Phi d_\eta(n_\Phi d_\eta - 1)\Phi_* \eta_*^{n_\Phi}(1-\xi)^{n_\Phi d_\eta - 2} \end{aligned} \tag{5.21}$$

By passing in Equations (5.20) and (5.21) to the limit as $\xi \to 1$, one infers that

$$\beta d_S T_*(S_*)^{n_\tau+1} = \eta_* \tag{5.22}$$

$$B^2 \beta d_S S_* = \tau_0 n_\Phi d_\eta(n_\Phi d_\eta - 1)\Phi_* \eta_*^{n_\Phi} \tag{5.23}$$

Hence,

$$S_* = \left[\frac{B^2}{\tau_0 n_\Phi d_\eta (n_\Phi d_\eta - 1)\Phi_* (\beta d_S)^{n_\Phi - 1} T_*^{n_\Phi}}\right]^{1/(n_\tau+1)n_\Phi - 1} \tag{5.24}$$

$$\eta_* = \left[\frac{B^{2(n_\tau+1)}(\beta d_S)^{n_\tau}}{T_*[\tau_0 n_\Phi d_\eta (n_\Phi d_\eta - 1)\Phi_*]^{n_\tau+1}}\right]^{1/(n_\tau+1)n_\Phi - 1} \tag{5.25}$$

Now, let us investigate the behavior of the functions f_S and f_η near $\xi = 0$. We assume the following expansions:

$$f_\eta(\xi) = 1 - f_\eta^* \xi^{e_\eta} + \cdots \tag{5.26}$$

$$f_S(\xi) = f_S(0) - f_S^* \xi^{e_S} + \cdots \tag{5.27}$$

$$\Phi(f_\eta(\xi)) = \Phi(1) - C\xi + \cdots \tag{5.28}$$

and

$$\Phi(S) = \Phi(1) - \Phi_{**}(1 - S)^{m_\Phi} + \cdots \tag{5.29}$$

The coefficients f_η^*, f_S^*, A and Φ_{**}, as well as the exponents m_Φ, e_S and e_η, are all positive numbers to be determined. Substituting Equations (5.26)–(5.29) into the system of equations (5.12) and (5.13) and accounting for Equation (5.16), one obtains the following relationships:

$$T_*(f_S(0) - f_S^* \xi^{e_S})^{n_\tau}\left[\alpha_S f_S(0) - f_S^* \xi^{e_S} + f_S^* \beta e_S \xi^{e_S}\right] = 1 - f_\eta^* \xi^{e_\eta} + \cdots \tag{5.30}$$

$$B^2(\alpha_S f_S(0) - f_S^* \xi^{e_S} + f_S^* \beta e_S \xi^{e_S} + \cdots) = \tau_0 \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\Phi(f_\eta(\xi)) \tag{5.31}$$

$$\Phi(1) - C\xi = \Phi(1) - \Phi_{**}(\eta_{**}\xi^{e_\eta})^{m_\Phi} + \cdots \tag{5.32}$$

From the last equation one immediately deduces that

$$e_\eta = \frac{1}{m_\Phi} \tag{5.33}$$

and

$$C = \Phi_{**}(f_\eta^*)^{m_\Phi} \tag{5.34}$$

From Equations (5.30) and (5.16) one infers that

$$e_S = e_\eta \tag{5.35}$$

and

$$T_* f_S(0)^{n_\tau} f_S^*(-1 + \beta e_S - n_\tau \alpha_S) = f_\eta^* \tag{5.36}$$

For a given function f_η, the function f_S can be calculated by solving the ordinary differential equation (5.12) with the terminal condition

$$f_S(1) = 0 \tag{5.37}$$

To simplify this calculation, put

$$w(\xi) = f_S(\xi)^{n_\tau + 1} \tag{5.38}$$

Then, from Equation (5.13), one obtains

$$T_* \left[\alpha_S w(\xi) - \frac{\beta}{n_\tau + 1} w'(\xi)\xi \right] = f_\eta(\xi) \tag{5.39}$$

The last equation can also be written as

$$w'(\xi)\xi = \frac{\alpha_S(n_\tau + 1)}{\beta} w(\xi) - \frac{n_\tau + 1}{T_* \beta} f_\eta(\xi) \tag{5.40}$$

From the terminal condition (5.37) one infers that

$$w(\xi) = \frac{1}{2T_* n_\tau} \xi^{2+2/n_\tau} \int_\xi^1 \frac{f_\eta(z)}{z^{3+2/n_\tau}} \, \mathrm{d}z \tag{5.41}$$

Thus, one obtains

$$f_S(\xi) = \frac{1}{(2T_* n_\tau)^{1/(n_\tau+1)}} \xi^{2/n_\tau} \left(\int_\xi^1 \frac{f_\eta(z)}{z^{3+2/n_\tau}} \, \mathrm{d}z \right)^{1/(n_\tau+1)} \tag{5.42}$$

Therefore, the actual instantaneous saturation of water at early times can be expressed through the effective saturation:

$$S(t, x) = \left(\frac{t/\tau_0}{2T_* n_\tau} \right)^{1/(n_\tau+1)} x^{2/n_\tau} \left(\int_x^{aB(t/\tau_0)^\beta} \frac{\eta(t, \varsigma)}{\varsigma^{3+2/n_\tau}} \, \mathrm{d}\varsigma \right)^{1/(n_\tau+1)} \tag{5.43}$$

Equations (5.42) and (5.43) characterize the relationship between effective and instantaneous saturations at early times. To find both profiles, one needs to find the solution to the system of ordinary differential equations (5.12) and (5.13) subject to the four boundary conditions (5.14), (5.15) and (5.17). To solve this system, one needs only three conditions. Nevertheless, the above-formulated boundary-value problem is not overdetermined because systems (5.12) and (5.13) also includes two unknown parameters: S_* and B. To determine these parameters, one can use one of the boundary conditions and Equation (5.24). Since the coefficients in front of the derivatives in Equations (5.12) and (5.13) vanish at the boundaries, the asymptotic

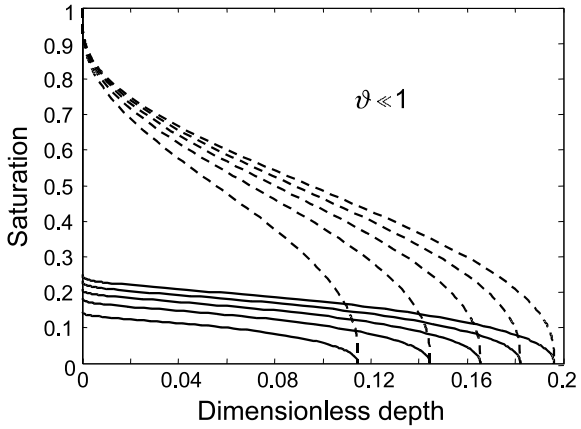

Figure 3. An example of calculation of self-similar solution (5.1) and (5.2): dashed lines – effective saturation, solid lines – instantaneous saturation (for $n_\tau = 2$). The dimensionless time ϑ is much less than 1.

representations (5.18), (5.19) and (5.26), (5.27) can be used to find the solution numerically.

An example of calculation of the wetting fluid saturation profiles using the self-similar solution (5.1) and (5.2) is presented in Figure 3. The saturation profile has the shape of an expanding tongue whose slope increases indefinitely both at the inlet face of the block and at the tip. In the calculations above we used a model function

$$\Phi(\eta) = \int_0^\eta 30S^3(1-S)^2\,\mathrm{d}S \tag{5.44}$$

cf. Equation (27) (Barenblatt and Gilman, 1987).

To estimate the volume of non-wetting fluid displaced by the advancing wetting fluid from a portion of the block of cross-sectional area A at early times, one needs to evaluate the integral of solution (5.1) between 0 and $S_0(t) = S(T_0(t))$:

$$R(t) = A\int_0^{x(t)} S(t,x)\,\mathrm{d}x = A\left(\frac{t}{\tau_0}\right)^{\alpha_S+\beta} B\sqrt{\frac{\sigma}{\mu_\mathrm{w}}\sqrt{\frac{k}{\varphi}}}\int_0^1 f_S(\xi)\,\mathrm{d}\xi \tag{5.45}$$

By virtue of Equations (5.8) and (5.9), oil recovery at early times has the following asymptotics:

$$R(t) \sim t^{1/2+1/2(1+n_\tau)} \tag{5.46}$$

If $n_\tau = 0$, that is, if the relaxation time does not depend on wetting fluid saturation, the volume of imbibed water and, respectively, oil recovery, is linear in time. This linear scaling is in agreement with the result reported earlier by Barenblatt and

Gilman (1987). In contrast, as $n_\tau \to \infty$, that is, the relaxation time becomes close to zero, the non-wetting fluid production by early-time imbibition is proportional to the square root of time. We conclude this section with the following remark regarding the magnitude of the time interval where neglecting the actual saturation in Equation (2.25) is admissible. Clearly, only the last term in Equation (5.11) depends on time. For example, if $n_\tau = 2$, then from Equations (5.8) and (5.9) $\alpha_S = \beta = 1/3$, that is, the last term in (5.11) is proportional to the cube root of time.

6. Asymptotic Solution at Large Times

In this section, we obtain an asymptotic solution to problem (2.25), (2.26) and (2.28), (2.33), valid at large times. Namely, we assume that on a large portion of the interval where the actual instantaneous saturation is non-zero, it exceeds the threshold above which the relaxation time is close to a constant value τ_0, see Figure 2. For such values of S, Equation (2.25) becomes

$$\tau_0 \frac{\partial S}{\partial t} = \eta - S \tag{6.1}$$

Let us seek a solution to the initial and boundary-value problem (2.25), (2.26) and (2.28), (2.33) in the form

$$S(t, x) = \varphi(\vartheta) f_S(\xi) \tag{6.2}$$

$$\eta(t, x) = f_\eta(\xi) \tag{6.3}$$

where ϑ and ξ are defined in Equations (4.1) and (4.2). Functions $\varphi(\vartheta)$, $X(\vartheta)$ (see Equation (4.7)), $f_S(\xi)$, and $f_\eta(\xi)$ are yet to be determined. The reason why we use the same similarity variable ξ for both instantaneous and effective saturations is the same as above: at $t > 0$, both saturations are either positive or equal to zero simultaneously. As above, $f_\eta(0) = 1$ by virtue of boundary condition (5.15). Clearly,

$$x(t) \to \infty \quad \text{as } t \to \infty \tag{6.4}$$

whereas function $\varphi(\vartheta)$ is uniformly bounded. Formal substitution of Equations (6.2) and (6.3) into Equations (2.25) and (2.26) yields

$$T(\varphi(\vartheta) f_S(\xi)) \left(\frac{\mathrm{d}\varphi(\vartheta)}{\mathrm{d}\vartheta} f_S(\xi) - \varphi(\vartheta) f_S'(\xi) \frac{\xi}{X(\vartheta)} \frac{\mathrm{d}X(\vartheta)}{\mathrm{d}\vartheta} \right) = f_\eta(\xi) - \varphi(\vartheta) f_S(\xi) \tag{6.5}$$

$$\frac{\mathrm{d}\varphi(\vartheta)}{\mathrm{d}\vartheta} f_S(\xi) - \varphi(\vartheta) f_S'(\xi) \frac{\xi}{X(\vartheta)} \frac{\mathrm{d}X(\vartheta)}{\mathrm{d}\vartheta} = \frac{\tau_0 a^2}{X(\vartheta)^2} \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \Phi(f_\eta(\xi)) \tag{6.6}$$

At later times, the interval of values of x, where the wetting fluid saturation is non-zero but still smaller than the threshold S_τ, above which the relaxation time is almost constant, is small in comparison with the wetting fluid penetration depth. Therefore, in an overwhelmingly large portion of the imbibed part of the porous block, the relaxation time is equal to τ_0 (see Figure 2). Taking this observation into account, one can rewrite Equation (6.5) in a slightly modified form

$$\left(\frac{\mathrm{d}\varphi(\vartheta)}{\mathrm{d}\vartheta} + \varphi(\vartheta)\right) f_S(\xi) + \varphi(\vartheta) f_S'(\xi) \frac{\xi}{X(\vartheta)} \frac{\mathrm{d}X(\vartheta)}{\mathrm{d}\vartheta} = f_\eta(\xi) \tag{6.7}$$

Furthermore, the rate of wetting fluid propagation decays in time. Therefore, due to (6.4), the ratio $(1/X(\vartheta))(\mathrm{d}X(\vartheta)/\mathrm{d}\vartheta)$ is negligibly small in comparison with two other terms in Equation (6.7). Hence, at large ϑ, Equation (6.5) reduces to

$$\left(\frac{\mathrm{d}\varphi(\vartheta)}{\mathrm{d}\vartheta} + \varphi(\vartheta)\right) f_S(\xi) = f_\eta(\xi) \tag{6.8}$$

Dividing both sides of the last equation by $f_S(\xi)$, one obtains

$$\frac{\mathrm{d}\varphi(\vartheta)}{\mathrm{d}\vartheta} + \varphi(\vartheta) = \frac{f_\eta(\xi)}{f_S(\xi)} \tag{6.9}$$

As the left-hand side does not depend on ξ and the right-hand side does not depend on ϑ, both must be constant. Clearly, this constant is positive. It can be easily demonstrated that the choice of this constant does not matter: by virtue of Equation (6.2), the saturation depends only on the product of φ and S. Therefore, whatever positive constant is selected, it cancels after the substitution into Equation (6.2). Hence, let us put this constant equal to 1, then

$$f_S(\xi) = f_\eta(\xi) \tag{6.10}$$

For consistency with the initial conditions (2.28) and (2.29), put

$$\varphi(\vartheta) = 1 - \mathrm{e}^{-\vartheta} \tag{6.11}$$

cf. Equation (3.6). Due to Equation (6.10), we can skip the subscripts S and η in $f_S(\xi)$ and $f_\eta(\xi)$ without confusion. Now, let us analyze Equation (6.6). The contribution of the first term $\mathrm{d}\varphi/\mathrm{d}\vartheta = \mathrm{e}^{-\vartheta}$ is negligibly small at large ϑ. Neglecting this term, one obtains from Equation (6.6):

$$\frac{A_S \varphi(\vartheta)}{\tau_0 a^2} \frac{\mathrm{d}(X(\vartheta))^2}{\mathrm{d}\vartheta} = \frac{(\mathrm{d}^2/\mathrm{d}\xi^2)\Phi(f(\xi))}{f'(\xi)\xi} \tag{6.12}$$

Just as for Equation (6.9), one infers that both sides of Equation (6.12) are equal to a non-negative constant, which we denote by Ξ^2. Then, for $X(\vartheta)$ one obtains an ordinary differential equation

$$\frac{\mathrm{d}(X(\vartheta))^2}{\mathrm{d}\vartheta} = \frac{\tau_0 a^2 \Xi^2}{1 - \mathrm{e}^{-\vartheta}} \tag{6.13}$$

whose solution is

$$X(\vartheta) = a\Xi\sqrt{\tau_0(\vartheta + \ln(1 - \mathrm{e}^{-\vartheta}))} \tag{6.14}$$

or, equivalently,

$$x(t) = a\Xi\sqrt{t + \tau_0 \ln(1 - \mathrm{e}^{-t/\tau_0})} \tag{6.15}$$

For the function $f(\xi)$ one gets

$$-\frac{\Xi^2}{2} f'(\xi)\xi = \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\Phi(f(\xi)) \tag{6.16}$$

where the coefficient Ξ has to be chosen in such a way that the three boundary conditions

$$f(0) = 1, \quad f(1) = \frac{\partial}{\partial\xi}\Phi(f(\xi))\bigg|_{\xi=1} = 0 \tag{6.17}$$

implied by Equations (2.29), (2.32) and (2.33), are fulfilled simultaneously. By analogy with a linear case, coefficient Ξ is often called a non-linear eigenvalue.

By a change of variable $\tilde{\xi} = \Xi\xi$, Equation (6.16) becomes identical to the one obtained by Ryzhik (1960). The only difference is that we look for a solution to Equation (6.16) on a fixed interval, whereas in Ryzhik (1960) the coefficient Ξ is fixed and the length of the interval where the solution is non-zero is an unknown parameter. Clearly, both the function $f(\xi)$ and the coefficient Ξ depend on the relative permeabilities and capillary pressure. For the particular function (5.44), the plot of the solution to boundary-value problem (6.16) and (6.17) is presented in Figure 4, and the obtained value of coefficient Ξ is equal to $\sqrt{2.0}$.

Summarizing the calculations above, one obtains the following expressions for the actual and effective saturations at later times:

$$S(t, x) = (1 - \mathrm{e}^{-t/\tau_0}) f\left(\frac{x}{a\Xi\sqrt{t + \tau_0 \ln(1 - \mathrm{e}^{-t/\tau_0})}}\right) \tag{6.18}$$

$$\eta(t, x) = f\left(\frac{x}{a\Xi\sqrt{t + \tau_0 \ln(1 - \mathrm{e}^{-t/\tau_0})}}\right) \tag{6.19}$$

The imbibition rate and the non-wetting fluid recovery can be calculated in a way similar to Equation (5.45):

$$R(t) = A\int_0^{x(t)} S(t, x)\,\mathrm{d}x = A\varphi\left(\frac{t}{\tau_0}\right) x(t) \int_0^1 f(\xi)\,\mathrm{d}\xi \tag{6.20}$$

Therefore,

$$R(t) = A(1 - \mathrm{e}^{-t/\tau_0}) a\Xi\sqrt{t + \tau_0 \ln(1 - \mathrm{e}^{-t/\tau_0})} \int_0^1 f(\xi)\,\mathrm{d}\xi \tag{6.21}$$

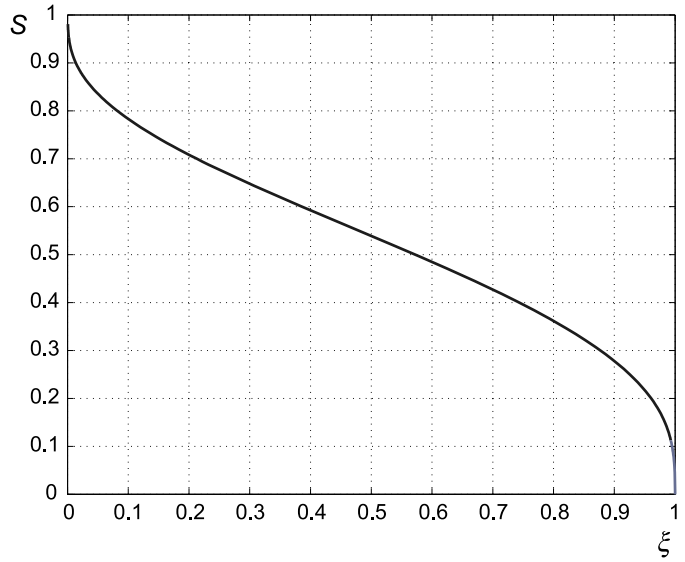

Figure 4. Solution to boundary-value problem (6.16) and (6.17).

For $\vartheta \gg 1$, Equation (6.21) admits a simplification because

$$\sqrt{\vartheta + \ln(1 - \mathrm{e}^{-\vartheta})} \approx \sqrt{\vartheta - \mathrm{e}^{-\vartheta}} \approx \sqrt{\vartheta} \tag{6.22}$$

and the logarithm can be neglected. Thus,

$$R(t) \approx V_0(1 - \mathrm{e}^{-t/\tau_0})\sqrt{\frac{t}{\tau_0}} \tag{6.23}$$

where

$$V_0 = a\,\Xi\sqrt{\tau_0}A \int_0^1 f(\xi)\,\mathrm{d}\xi \tag{6.24}$$

In other words, at a later time, the recovery is approximately proportional to the square root of time, that is, to the time scale suggested by the classical model of Ryzhik *et al.* (1961). An empirical coefficient with an exponential function in front of the radical in Equation (6.23) was earlier used to match experimental data, see, for example, Ma *et al.* (1997). The coefficient V_0 has the dimension of volume. Through the function f, which, in turn, is the solution of the boundary-value problem (6.16) and (6.17), V_0 depends on all parameters of the problem: the fluid viscosities, the relative permeability functions, etc.

To verify the model described above, we used the laboratory data kindly provided to us by Prof A. R. Kovscek, see Zhou *et al.* (2001). Experiments were performed with cores from the near-surface diatomite formation in Lompoc, California. The cores were cut in a direction parallel to the bedding plane, and shaped into cylinders with diameters of 2.5 cm and length of 9.5 cm. The porosity

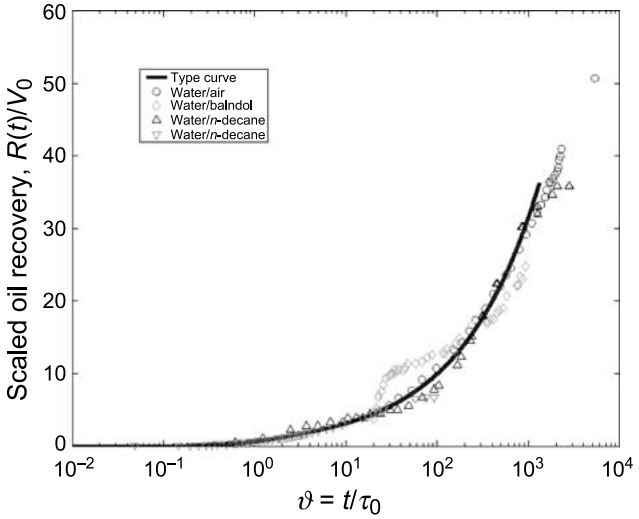

Figure 5. Matching scaled oil recovery. Data reported in Zhou *et al.* (2001).

was measured to be about 70%, and the absolute permeability was about 6 mD. For the imbibition experiments, the cores were dried and oil was pumped into the pore space. After fully saturating the core with oil, water was pumped through an endcap while the other endcap was sealed. The oil recovered by countercurrent imbibition was removed by the flowing water.

We have matched the measured fractional oil recovery versus time. Our fitting parameters were the relaxation time τ_0 and the coefficient V_0, see Equations (6.23) and (6.24). In Figure 5, the data points from different experiments reported by Zhou *et al.* (2001) are collapsed into a single curve generated by the dimensionless solution (6.23). The relaxation time τ_0 was estimated at about 8 s for imbibing a core filled with air, at about 220 s for a core filled with blandol and at approximately 1230 s for decane. Dimensionless ratio $\phi V / V_0$, where V is the total volume of the diatomite core, was estimated at 51.2, 47.0 and 10.6, respectively.

Clearly, Equation (6.23) becomes invalid as oil recovery approaches its maximum and the advancing water approaches the opposite end-surface of the core.

7. Boundary-Layer Type Solution

In this section, we outline another approach to solving systems (2.25) and (2.26). We assume that the character of dependence of relaxation time $\tau(S)$ on S is as shown in Figure 2. As the imbibing wetting fluid reaches a particular location, its saturation quickly overcomes the threshold after which the relaxation time can be assumed constant. To characterize the solution near the tip of the propagating water saturation profile, we employ a method similar to the one developed by Barenblatt *et al.* (1997a). The dynamics of the rest of the profile is described by the model developed by Barenblatt and Gilman (1987). To combine the two solutions, specific

boundary conditions are required. If $\tau(S) \equiv \tau_0$ is constant, then the system of Equations (2.25) and (2.26) can be reduced to a single partial differential equation of third order:

$$\frac{\partial \eta}{\partial t} = a^2 \frac{\partial^2}{\partial x^2} \Phi(\eta) + \tau_0 a^2 \frac{\partial^3}{\partial t\, \partial x^2} \Phi(\eta) \tag{7.1}$$

see Barenblatt and Gilman (1987). The initial condition $\eta_0(x)$ can be found from the equation

$$\eta_0(x) = \tau_0 a^2 \frac{\partial^2}{\partial x^2} \Phi(\eta_0(x)) \tag{7.2}$$

Since at the inlet of the block is contacted by the wetting fluid, at $\xi = 0$ one obtains the boundary condition

$$\eta(t, 0) = 1 \tag{7.3}$$

At the right end-point one formally has

$$\eta(t, \infty) = 0 \tag{7.4}$$

However, in fact, the solution vanishes at a finite $x(t)$ where the coordinate of the tip of the wetting fluid saturation profile propagates with the velocity

$$v(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} \tag{7.5}$$

Moreover, as the value of the function $\eta(t, x)$ approaches zero, the function $S(t, x)$ vanishes as well. If we denote the smallest wetting fluid saturation where $\tau(S) = \tau_0$ by δ, then the boundary condition (7.4) can be replaced with

$$\eta(t, x_\delta(t)) = \delta \tag{7.6}$$

where $x_\delta(t)$ is the distance from the inlet where the effective saturation is equal to δ. Thus, in Equation (7.6) the tip velocity (7.5) is related to the threshold saturation propagation, rather than to the tip of the profile. To be able to determine $x_\delta(t)$, one needs an additional boundary condition. To obtain such a condition at $x = x_\delta(t)$, let us replace the partial derivative $\partial S/\partial t$ with respect to t in the mass balance Equation (2.8) by the right-hand side of Equation (2.26). By integrating the result with respect to x and using Equation (2.23), one obtains

$$\phi a^2 \frac{\partial}{\partial x} \Phi(\eta) + u_w = 0 \tag{7.7}$$

The flux at the threshold saturation level is a function of the advancing fluid velocity:

$$\frac{\partial}{\partial x} \Phi(\eta) \bigg|_{x = x_\delta(t)} = F(v(t)) \tag{7.8}$$

If the wetting fluid flux at the threshold saturation is negligibly small, that is, $F(v) \cong 0$, condition (7.8) reduces to

$$\frac{\partial}{\partial x}\Phi(\eta)\bigg|_{x=x_\delta(t)} = 0 \tag{7.9}$$

and we end up with the initial and boundary-value problem, which was considered by Barenblatt and Gilman (1987).

To conclude this section, we remark that Zimmerman and Bodvarsson (1989) applied a boundary-layer method for obtaining an approximate solution to the problem of imbibition in unsaturated media in the classical formulation.

8. Conclusions

Barenblatt's model of non-equilibrium countercurrent imbibition has been investigated. We considered the spontaneous imbibition of a wetting fluid into a porous block initially saturated with a non-wetting fluid. The model is based on the approach proposed in Barenblatt (1971) and Barenblatt and Vinnichenko (1980). Namely, to characterize the modification of relative permeabilities and capillary pressure in spontaneous imbibition, an effective water saturation and a relaxation time are introduced. The main feature of the approach proposed in the present work is the assumption that the relaxation time is a function of the instantaneous water saturation. As the water saturation approaches its minimum value, the relaxation time approaches zero. Conversely, as the water saturation approaches its maximum value, the relaxation time goes to infinity.

Two asymptotic solutions have been obtained. The first one corresponds to early times, when water saturation is still close to zero throughout the porous block, including a domain next to the inlet face. At later time, the water saturation profile approaches the shape obtained with the classical model (Ryzhik *et al.*, 1961). Therefore, the result obtained numerically by Barenblatt and Gilman (1987) has been confirmed analytically. The time dependence of the recovered oil volume does not follow the square root of time scaling rule suggested by the classical models. Instead, it follows the pattern observed in numerous experiments. In particular, the data provided to us by Kovscek (Zhou *et al.*, 2001), obtained on similar core samples for different pairs of fluids, collapse to a single dimensionless curve derived from the model considered in the present work. Finally, a derivation of the boundary condition at the tip of water saturation profile based on the boundary-layer method has been outlined.

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