HYPERBOLIC SYSTEMS OF THREE-PHASE FLOW IN POROUS MEDIA

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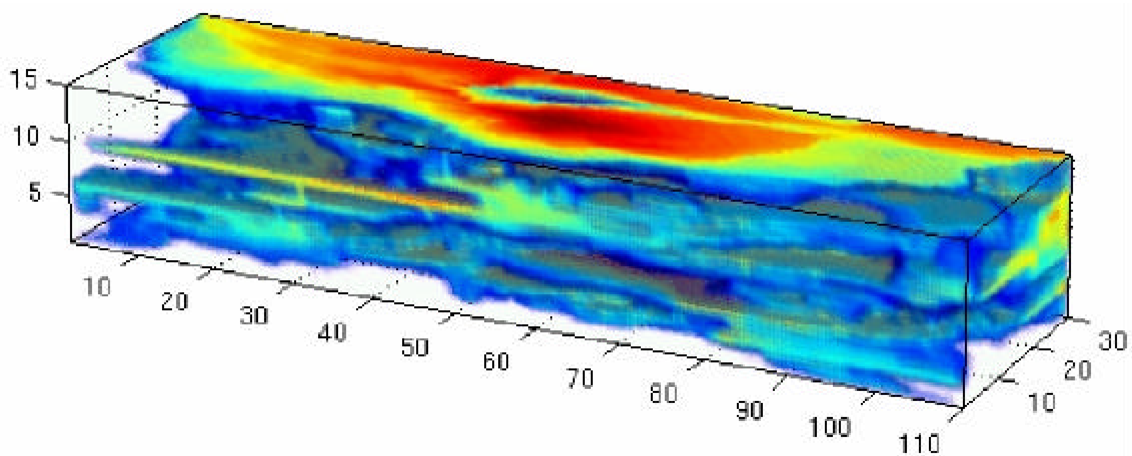

Gordon Conference on Flow and Transport in Permeable Media

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WHY THREE-PHASE FLOW

- **Three-phase flow is the rule, not the exception, in recovery processes in oil and gas reservoirs**
  - Primary production (gasdrive or waterdrive)
  - Secondary and tertiary production (waterflooding, steamflooding, CO_2 injection, ...)
- **Contamination of the shallow subsurface by hydrocarbon compounds**
- **Geological CO_2 sequestration**
  - Depleted oil and gas reservoirs
  - Unminable coal beds
  - Deep (saline) aquifers

WHAT DO WE PROPOSE, AND WHY

- **Objective:**
  - Exceptionally accurate, fast numerical solutions to realistic **three-phase flows** in porous media
- **Approach:**
  - Develop **analytical solution** to the Riemann problem
  - Use it as a building block for general 1D problems, via a **front-tracking** method
  - Solve three-phase flow along **streamlines**

MATHEMATICAL MODEL

- **Assumptions:**
  - Immiscible, incompressible fluids
  - Multiphase extension of Darcy's law
  - Negligible capillary effects

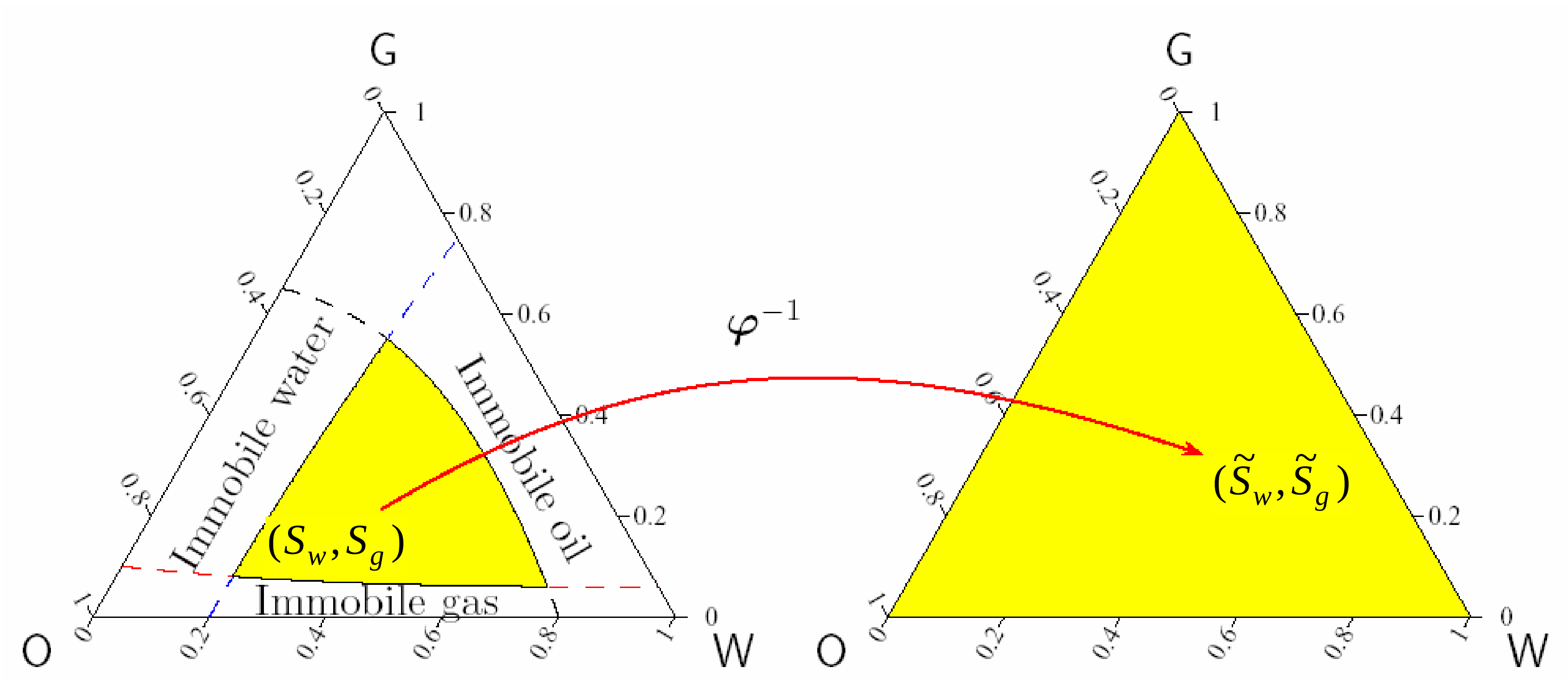
- **Equations:**
  - *Pressure* equation (elliptic)

$$\nabla \cdot \mathbf{v}_T = 0, \qquad \mathbf{v}_T = -l_T \frac{\mathbf{k}}{f} \nabla p, \qquad l_T \equiv l_w + l_o + l_g$$

  - A system of *saturation equations* (hyperbolic)

$$\partial_t \begin{pmatrix} S_w \\ S_g \end{pmatrix} + \mathbf{v}_T \cdot \nabla \begin{pmatrix} f_w \\ f_g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad f_w = l_w / l_T, \qquad f_g = l_g / l_T$$

THE SATURATION SPACE



CHARACTER OF THE SYSTEM

- The character of the system of first-order equations

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} + \partial_x \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Leftrightarrow \quad \partial_t \mathbf{u} + \partial_x \mathbf{f} = \mathbf{0}$$

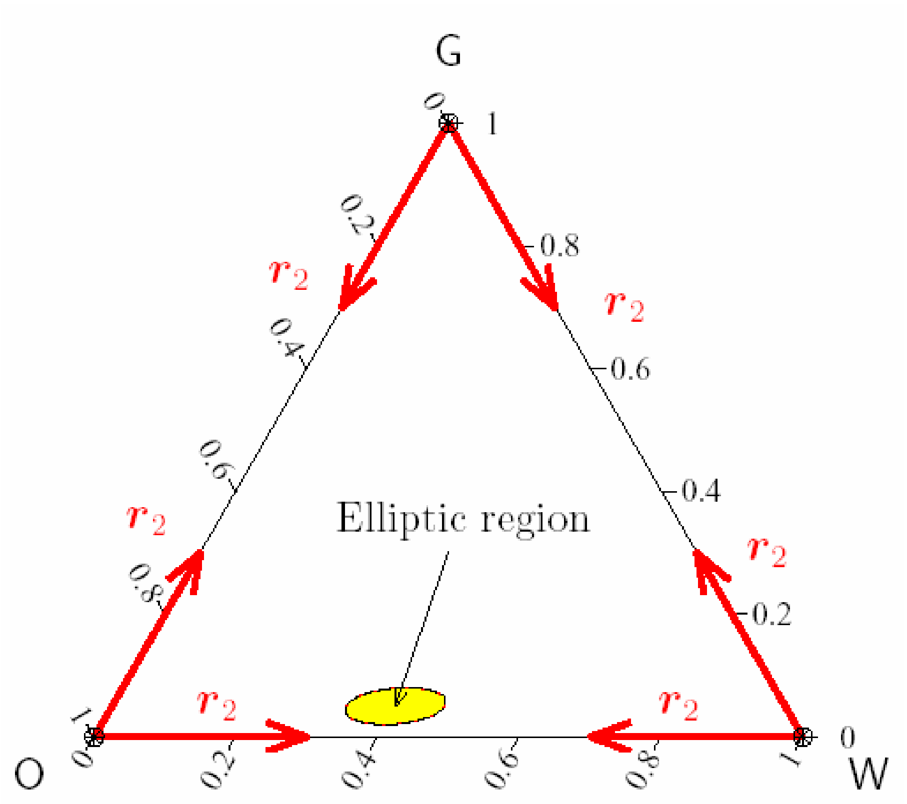
is determined by the eigenvalues ($\mathbf{n}_1$, $\mathbf{n}_2$) and eigenvectors ($\boldsymbol{r}_1, \boldsymbol{r}_2$) of the Jacobian matrix

$$\mathbf{f}'(\mathbf{u}) = \begin{pmatrix} f_{,u} & f_{,v} \\ g_{,u} & g_{,v} \end{pmatrix}$$

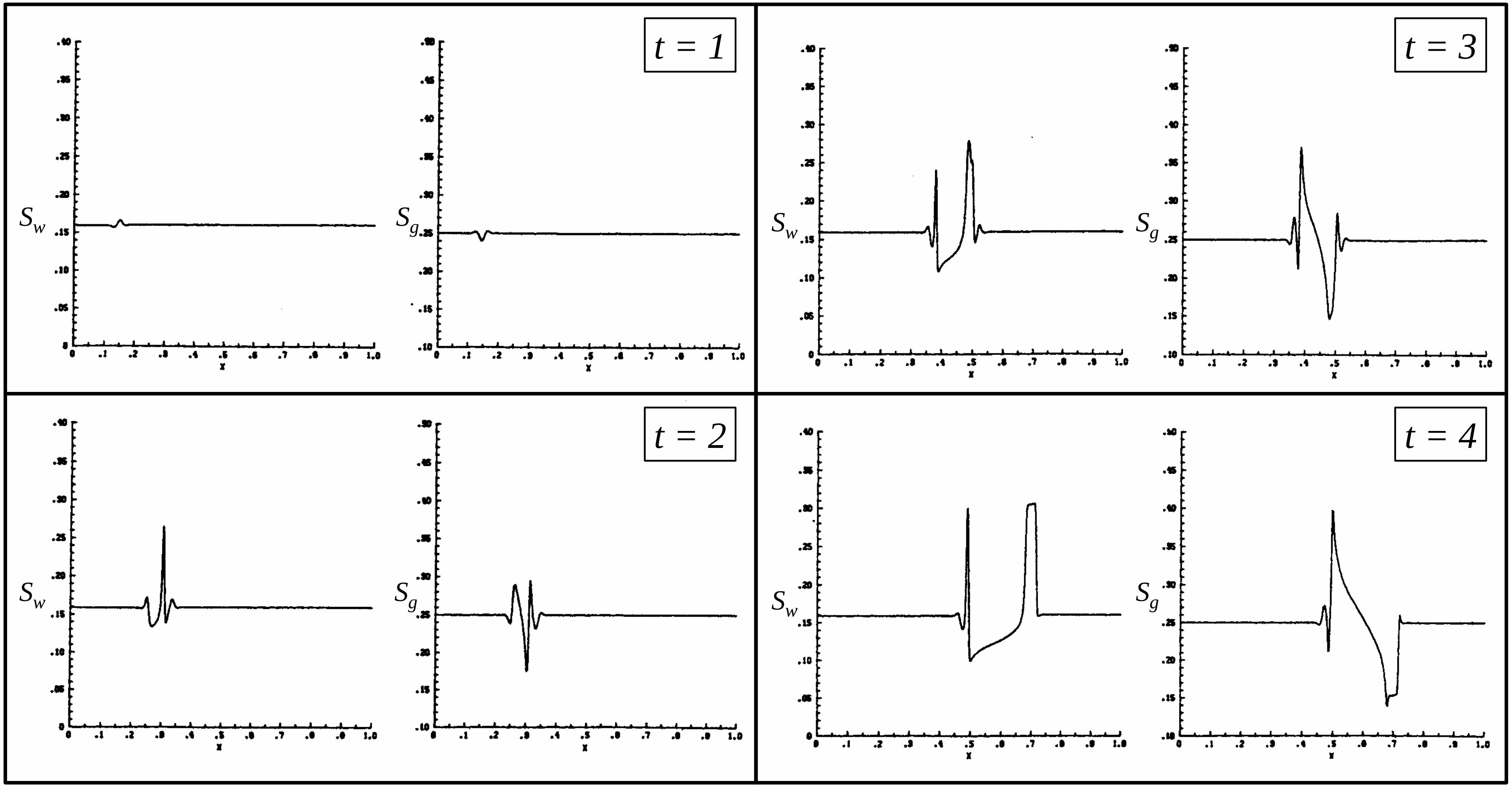
- **Hyperbolic:** the eigenvalues are real and
  the Jacobian matrix is diagonalizable
  - **Strictly hyperbolic:** eigenvalues are distinct, $\mathbf{n}_1 < \mathbf{n}_2$
- **Elliptic:** eigenvalues are complex conjugates

ELLIPTIC REGIONS

Regions in the saturation triangle, where the system of equations is **elliptic** rather than **hyperbolic**



CONSEQUENCES OF ELLIPTIC REGIONS



(Bell et al., 1986)

CONDITIONS FOR HYPERBOLICITY

- **Traditional approach:**

  Assume certain behavior of the relative permeabilities → *Infer* loss of hyperbolicity

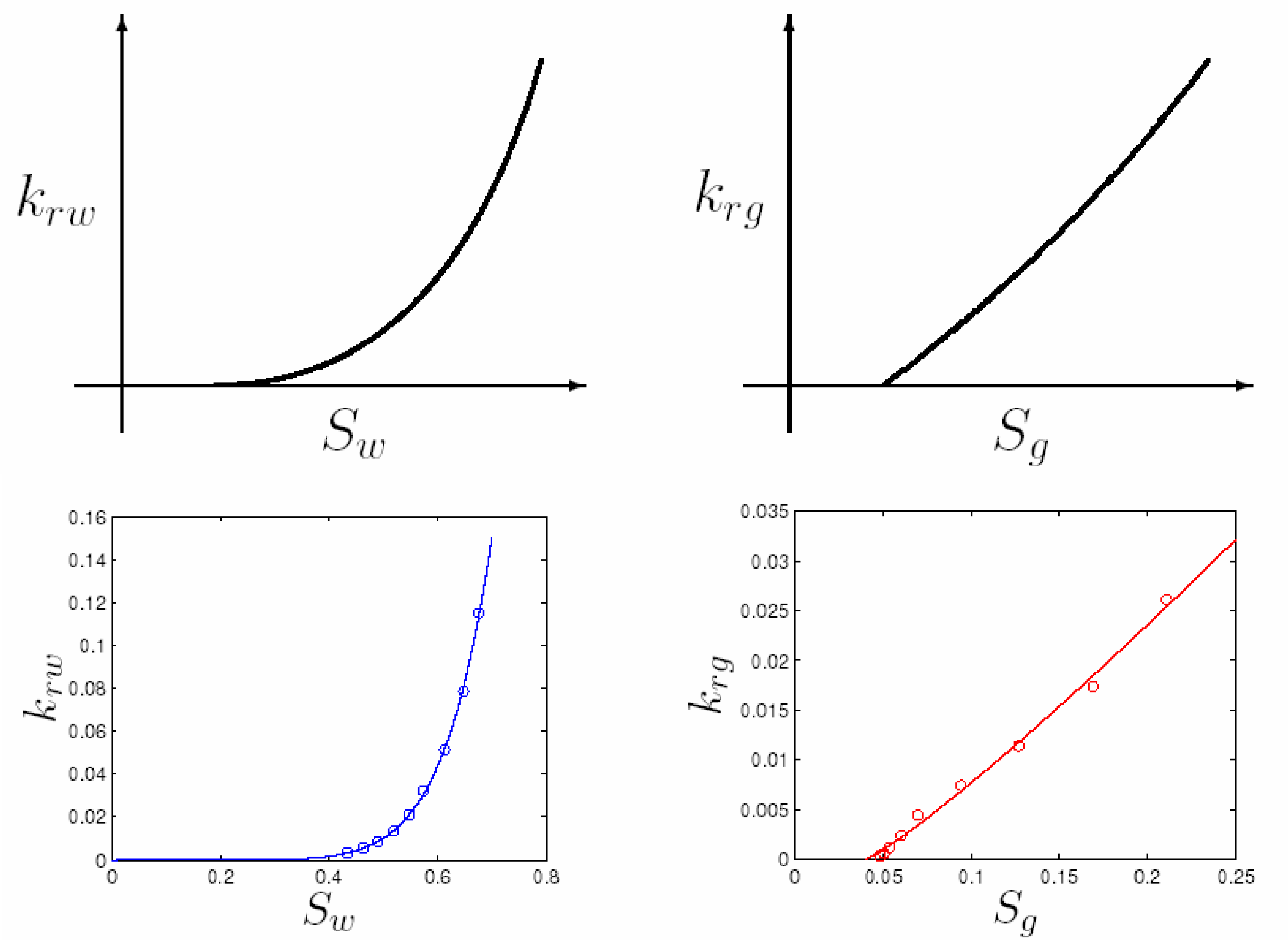
- **We propose a new approach:** (J.: *PhD* 2003) (J. and Patzek: *SPEJ* in press)

  Infer conditions on the relative permeabilities ← *Enforce* hyperbolicity

CONDITIONS FOR HYPERBOLICITY

(J. and Patzek: *TIPM* in press)

- **Essential condition:** a positive endpoint slope of the relative permeability of the least wetting phase



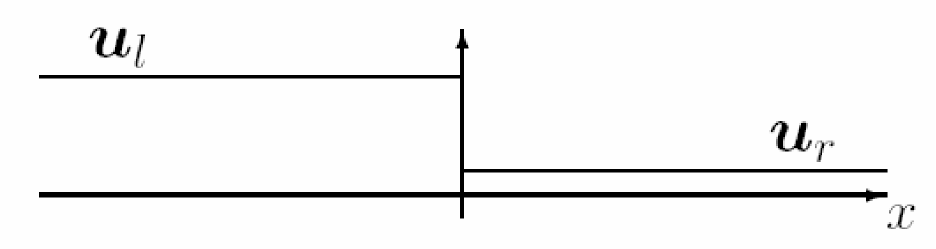
THE THREE-PHASE RIEMANN PROBLEM

- **Riemann problem:** find a weak (possibly discontinuous) solution to the 2×2 system of equations

$$\partial_t \mathbf{u} + \partial_x \mathbf{f} = \mathbf{0}, \quad -\infty < x < \infty,\ t > 0$$

with piecewise constant initial condition

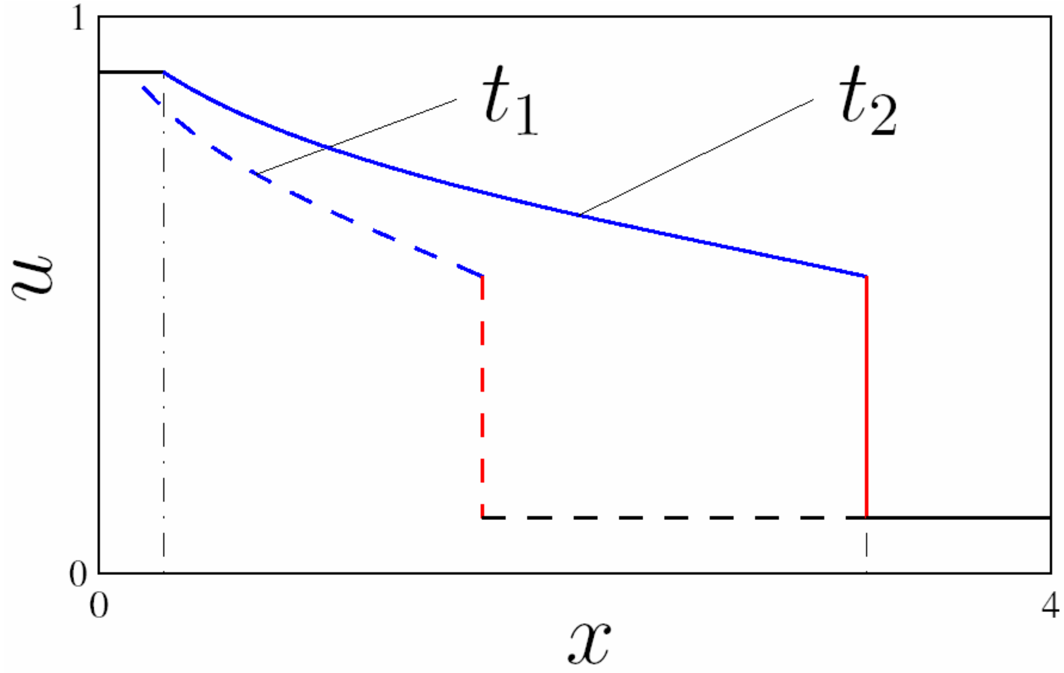
$$\mathbf{u}(x,0) = \begin{cases} \mathbf{u}_l, & x < 0 \\ \mathbf{u}_r, & x \geq 0 \end{cases}$$



- **Limitations of previous solutions:**
  - Sequence of two successive *two-phase flow* displacements
  - Saturation paths restricted to *straight lines*

SOLUTION TO THE RIEMANN PROBLEM

- **Self-similarity** (“stretching” or “coherence” principle)

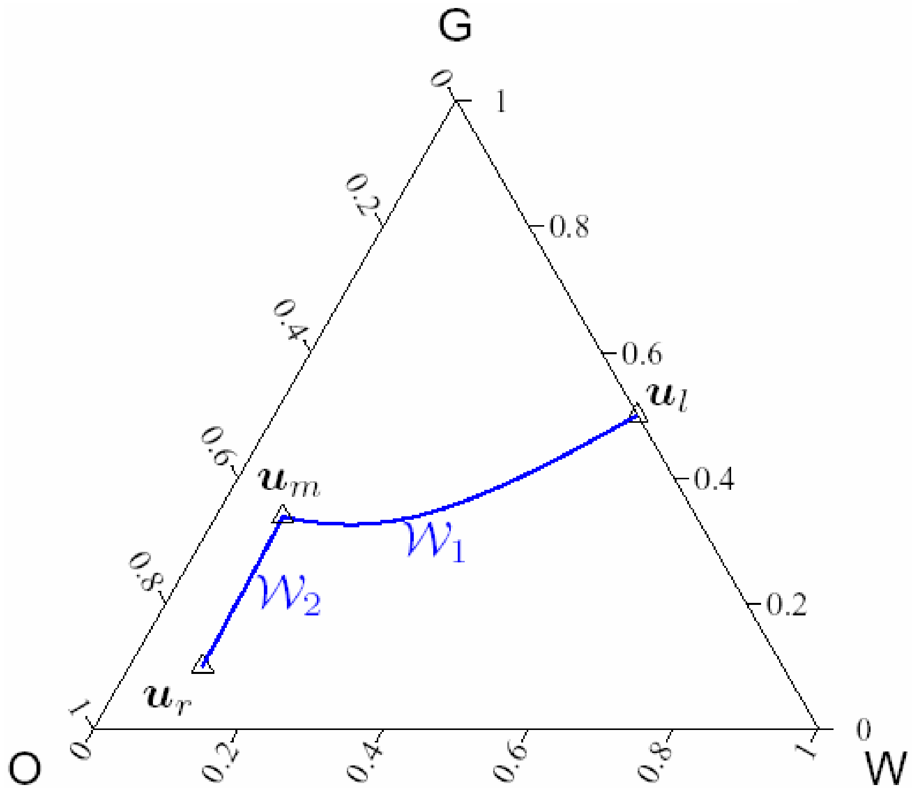


$$\mathbf{u}(x,t) = \mathbf{U}(V),$$

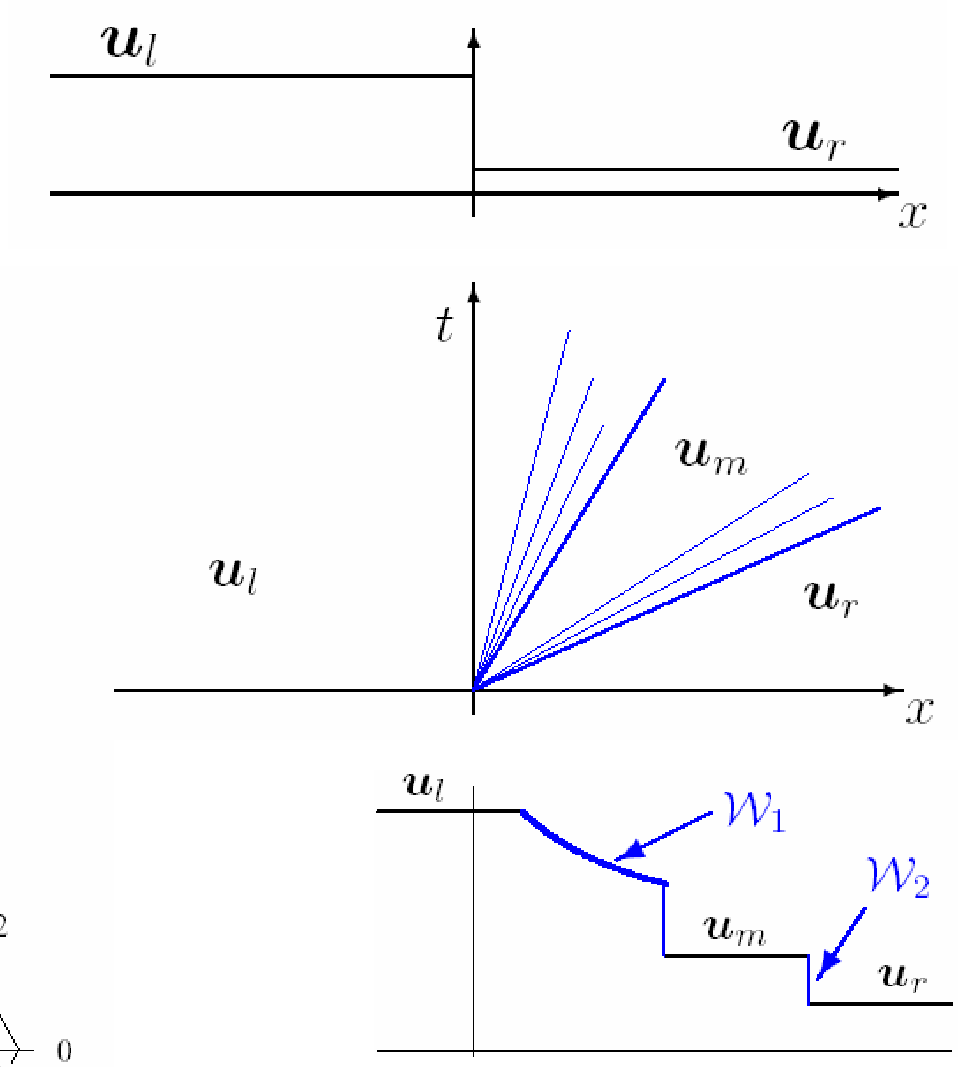
where $V = x/t$

- **Strict hyperbolicity** $\Rightarrow$ **Wave separation**

$$\boldsymbol{u}_l \xrightarrow{\mathcal{W}_1} \boldsymbol{u}_m \xrightarrow{\mathcal{W}_2} \boldsymbol{u}_r$$



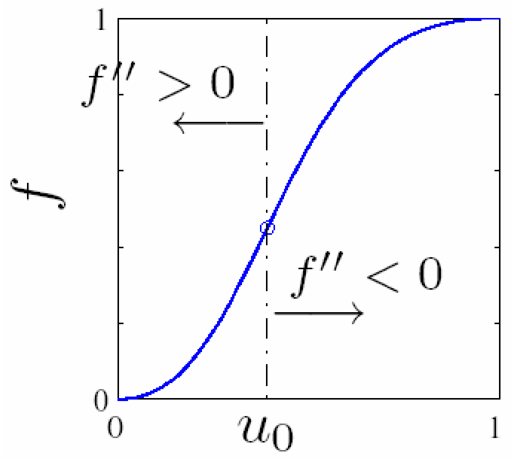
G
O
W
0
1
0.2
0.4
0.6
0.8
$\boldsymbol{u}_l$
$\boldsymbol{u}_m$
$\boldsymbol{u}_r$
$\mathcal{W}_1$
$\mathcal{W}_2$



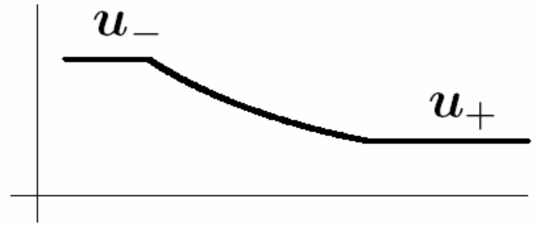
$\boldsymbol{u}_l$
$\boldsymbol{u}_r$
x
t
$\boldsymbol{u}_m$
$\mathcal{W}_1$
$\mathcal{W}_2$

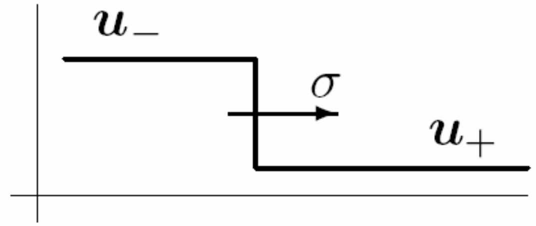
WAVE TYPES (TWO-PHASE FLOW)

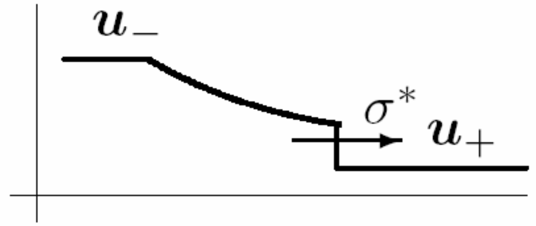
- The fractional flow function is S-shaped, with a single inflection point



- The only **admissible wave types** are:
  - Rarefaction ($\mathcal{R}$)
  - Shock ($\mathcal{S}$)
  - Rarefaction-shock ($\mathcal{RS}$)

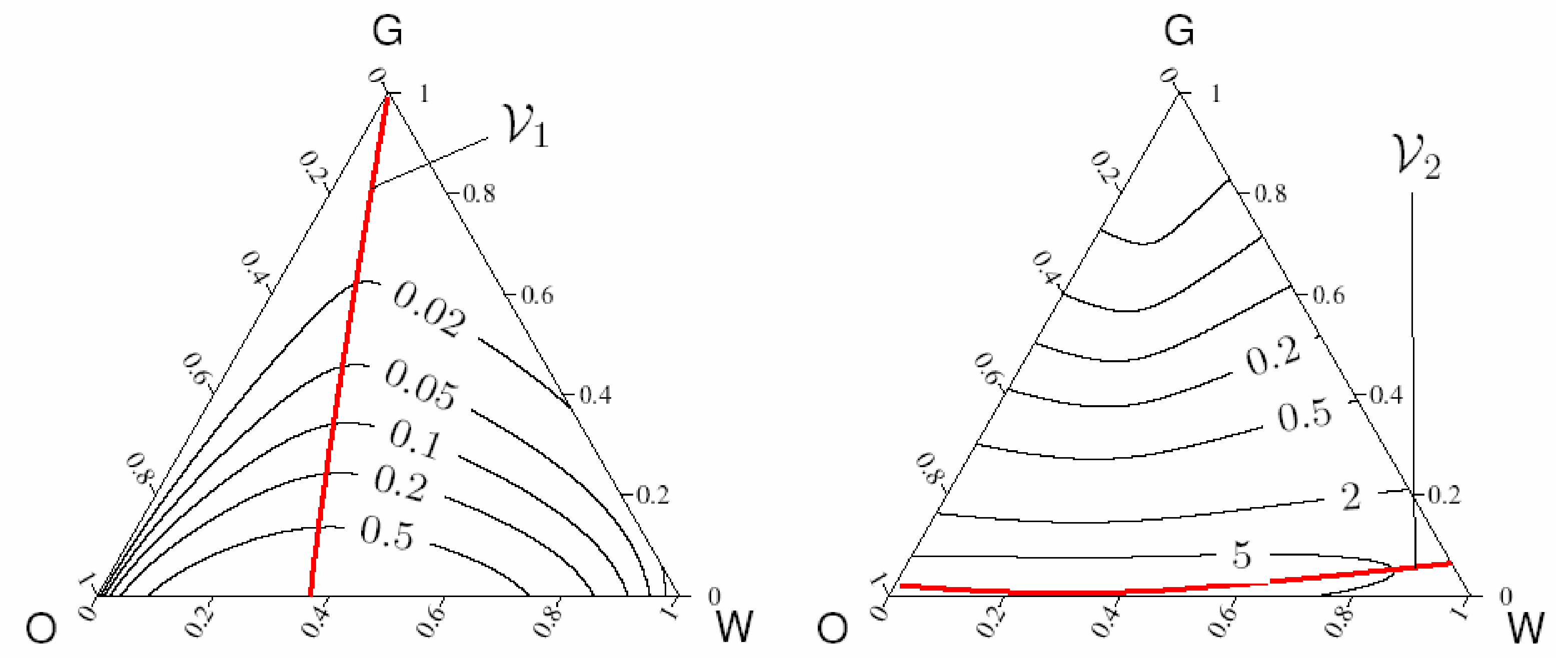






WAVE TYPES (THREE-PHASE FLOW)

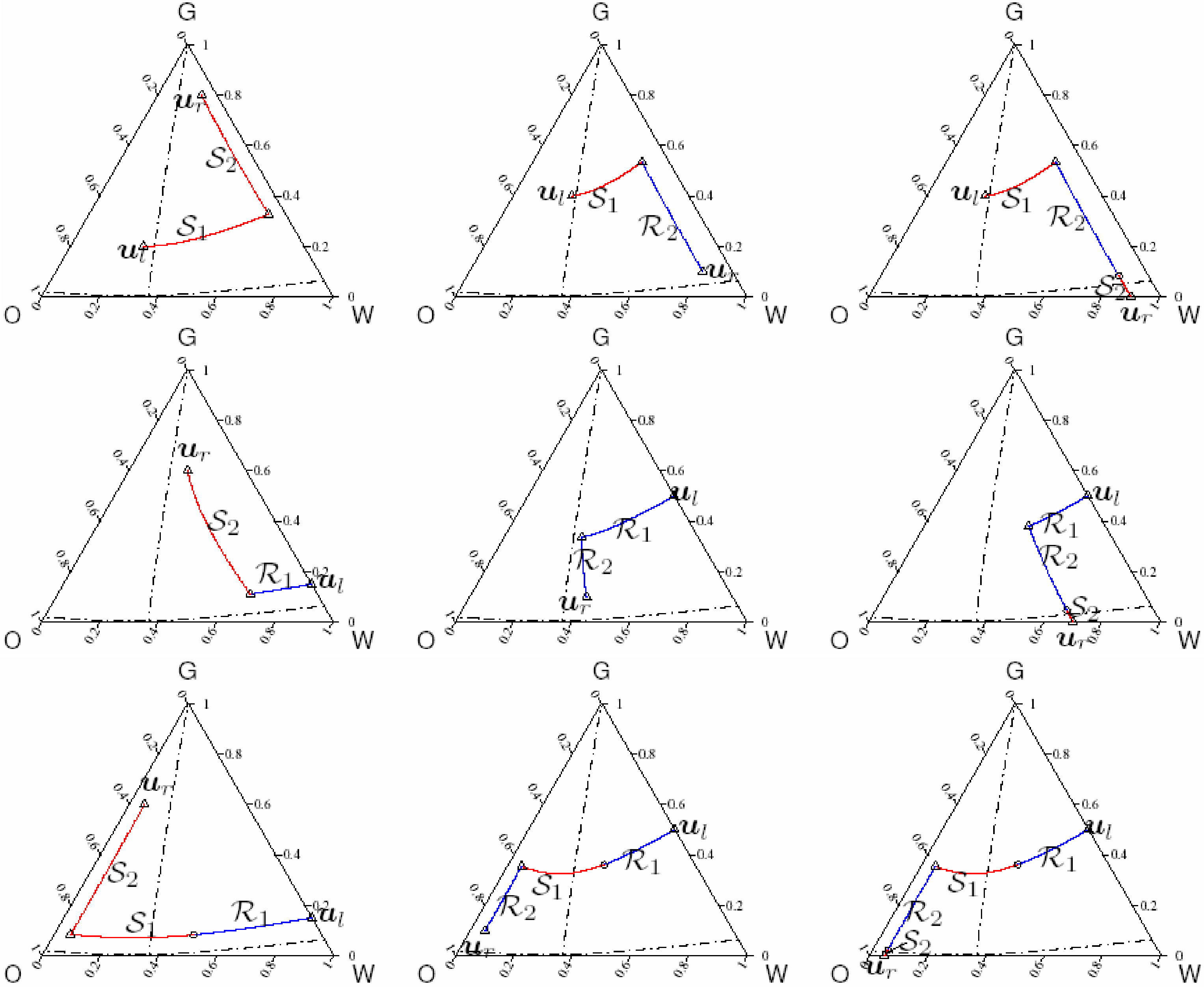
- The fractional flow functions have single, continuous inflection loci (natural generalization of the two-phase case)



- There are **9 admissible wave combinations**
  - Two separate waves: $\mathcal{W}_1$, $\mathcal{W}_2$
  - Each wave may only be of type $\mathcal{R}$, $\mathcal{S}$, or $\mathcal{RS}$

COMPLETE CATALOGUE OF SOLUTIONS

(J. and Patzek: *TIPM* 2004)



RIEMANN SOLVER ALGORITHM

1. Given injected (left) and initial (right) states: u_L, u_R

2. Set initial guess and trial solution: u_M^{tr}, $\mathcal{W}_1^{\mathrm{tr}} = \mathcal{R}_1$, $\mathcal{W}_2^{\mathrm{tr}} = \mathcal{R}_2$

3. Solve trial configuration and update wave structure:

$$[u_M, \mathcal{W}_1, \mathcal{W}_2] = \mathrm{WaveStruct}\,(u_L, u_R, u_M^{\mathrm{tr}}, \mathcal{W}_1^{\mathrm{tr}}, \mathcal{W}_2^{\mathrm{tr}})$$

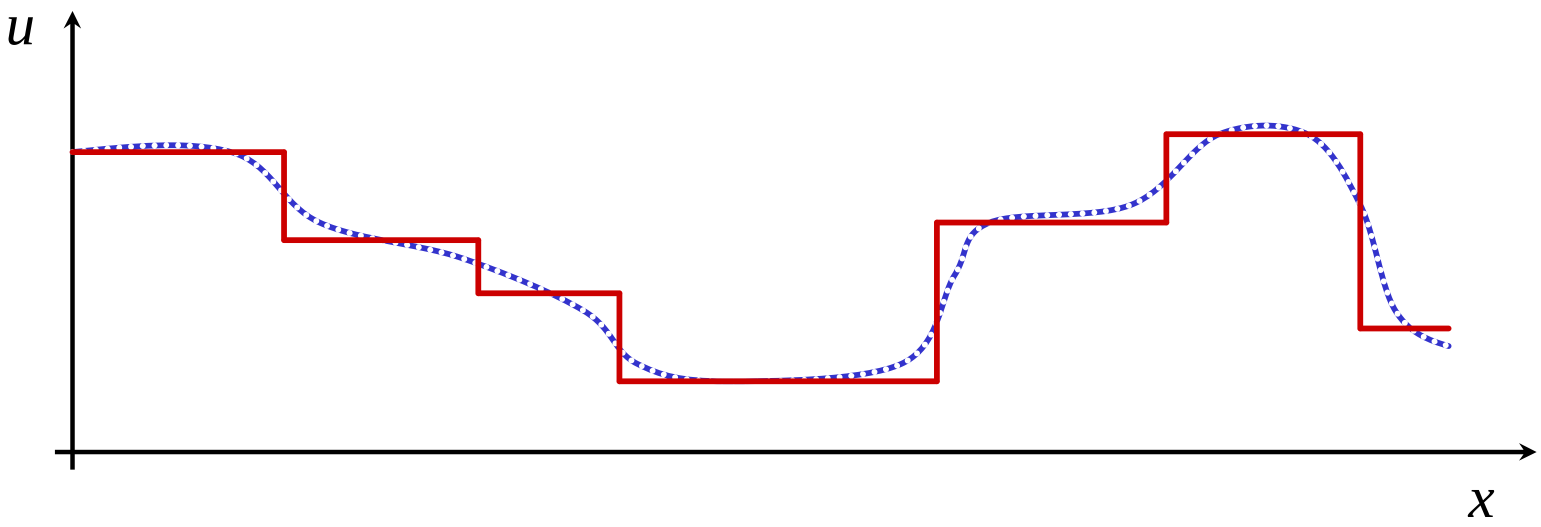
(J. and Patzek: *TIPM* 2004)

4. Check admissibility:
   If $(s_1 > s_2)$ { Set new initial guess: u_M
   Declare solution invalid: $\mathcal{W}_1^{\mathrm{tr}} = \mathcal{W}_2^{\mathrm{tr}} = 0$ }

5. Check convergence:
   If $\mathcal{W}_1\mathcal{W}_2 = \mathcal{W}_1^{\mathrm{tr}}\mathcal{W}_2^{\mathrm{tr}}$ Stop
   Else Set $\mathcal{W}_1^{\mathrm{tr}}\mathcal{W}_2^{\mathrm{tr}} \leftarrow \mathcal{W}_1\mathcal{W}_2$, $u_M^{\mathrm{tr}} \leftarrow u_M$, Goto 3.

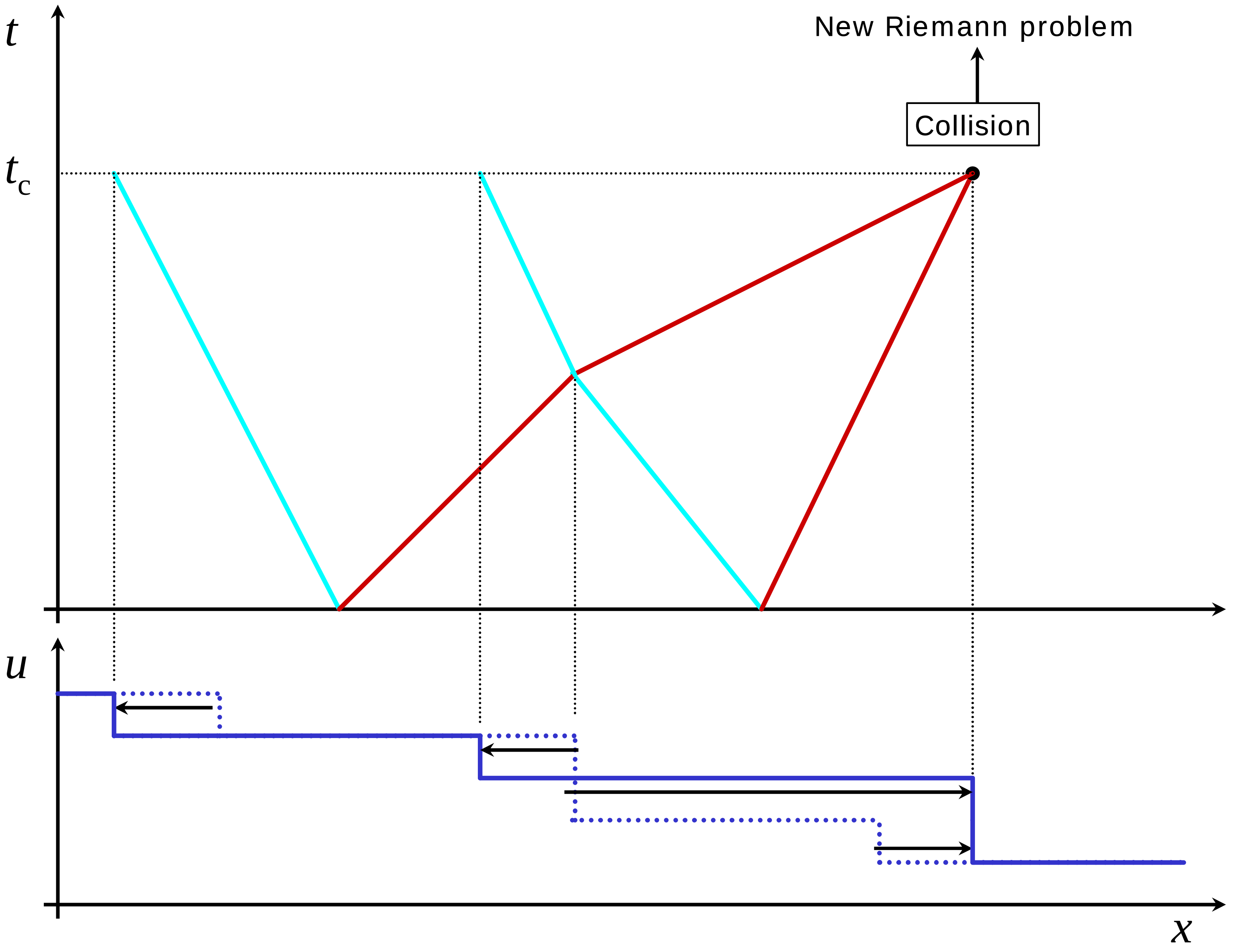
THREE-PHASE CAUCHY PROBLEM

- Solution to the Riemann problem is insufficient if:
  - Initial conditions different from constant
  - Variable injection saturations (e.g. WAG)



- **Front-tracking method:**
  - Piecewise constant approximation of the solution
  - Sequence of Riemann problems

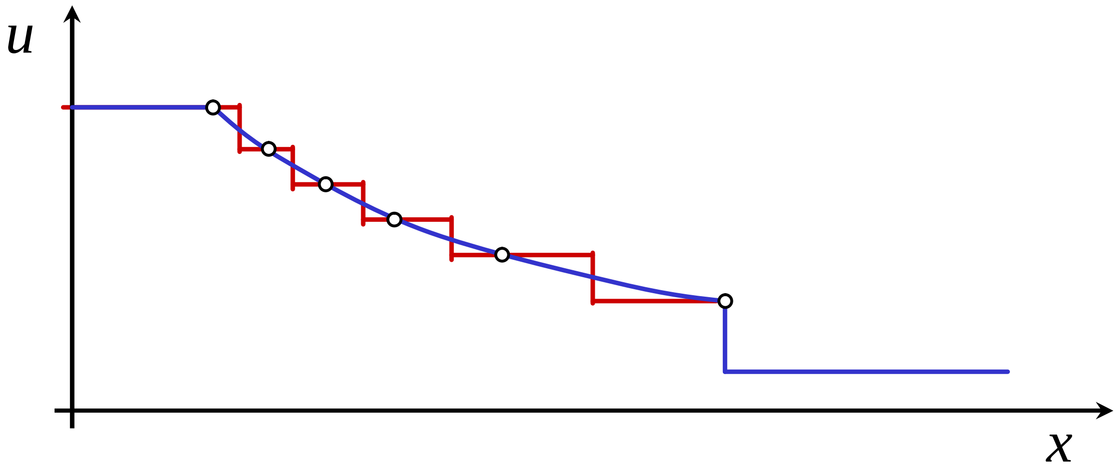
FRONT-TRACKING ALGORITHM



FRONT-TRACKING IMPLEMENTATION

- If the solution involves discontinuities only, the front-tracking method is **exact**

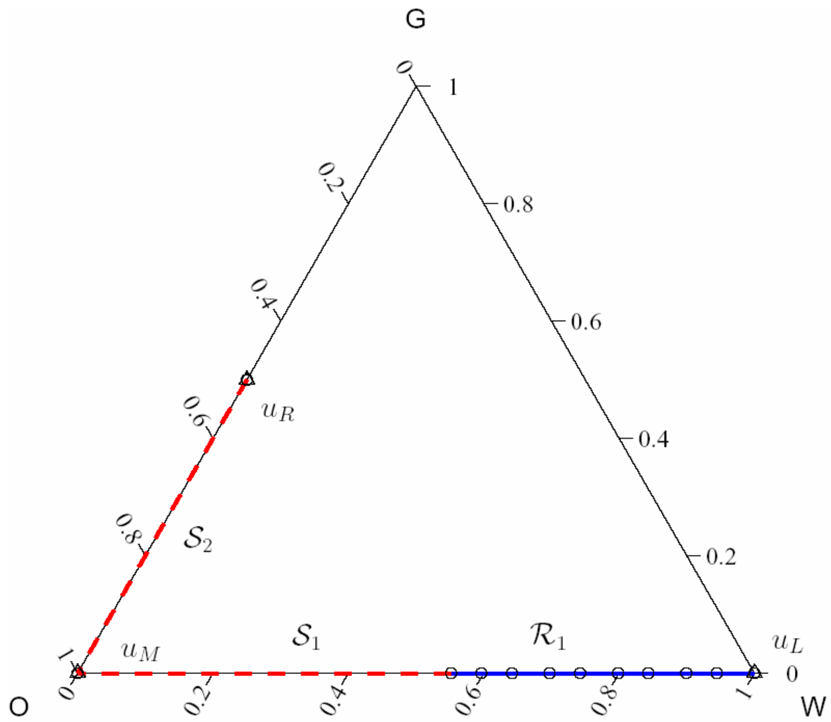
- **Rarefactions** are approximated by a series of (small) jump discontinuities



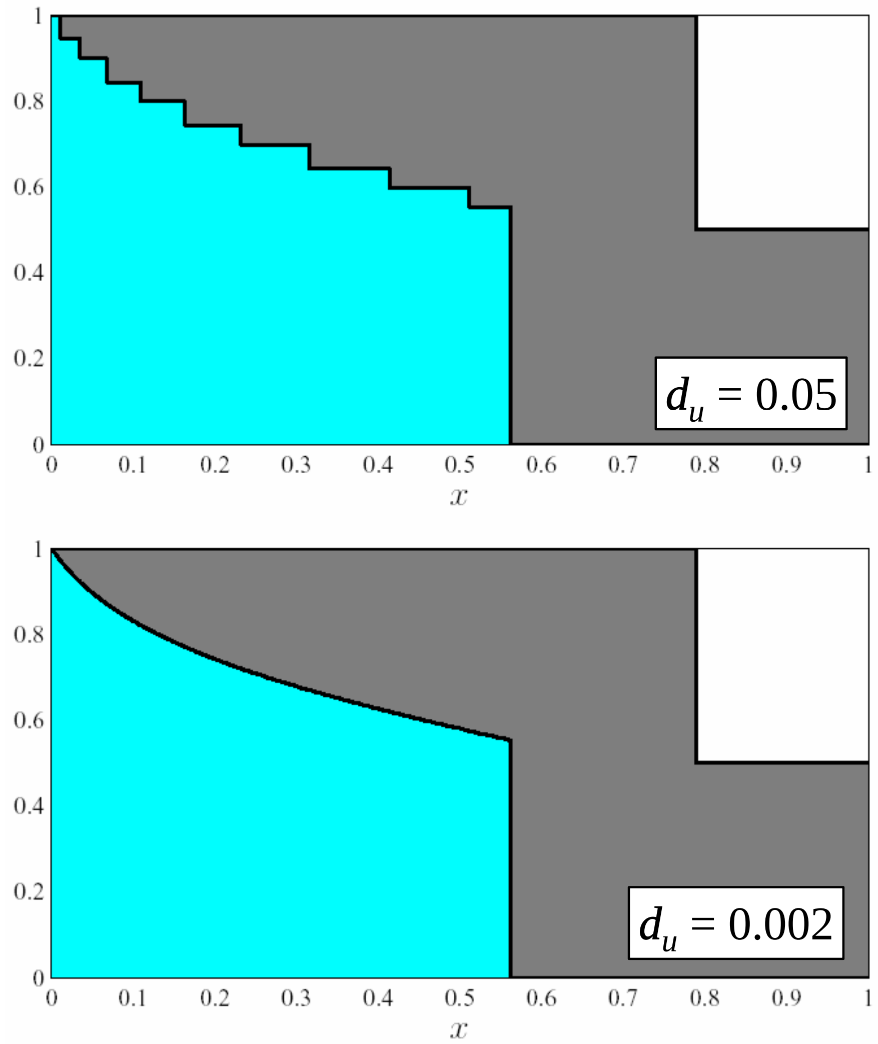
- **Data reduction:** Exceedingly small Riemann problems are discarded to avoid blow-up of number of discontinuities

EXAMPLE 1

- Riemann problem involving **local** wave curves



Saturation path:
Exact solution and front-tracking solution



EXAMPLE 2: LINEAR WAG

- Initially, reservoir with 80% oil, 20% gas
- Alternate cycles of water and gas injection
- Front-tracking solution with d_u = 0.005
- Half a million Riemann solves ~ 5 sec on desktop PC

STREAMLINE METHODS

- **Basic idea:** decouple the three-dimensional transport into a series of 1D problems along streamlines

- **Sequential solution** of pressure and saturations (IMPES)
  - Pressure equation (fixed saturations)

$$\nabla \cdot \mathbf{v}_T = 0, \quad \mathbf{v}_T = -l_T \frac{\mathbf{k}}{f} \nabla p$$

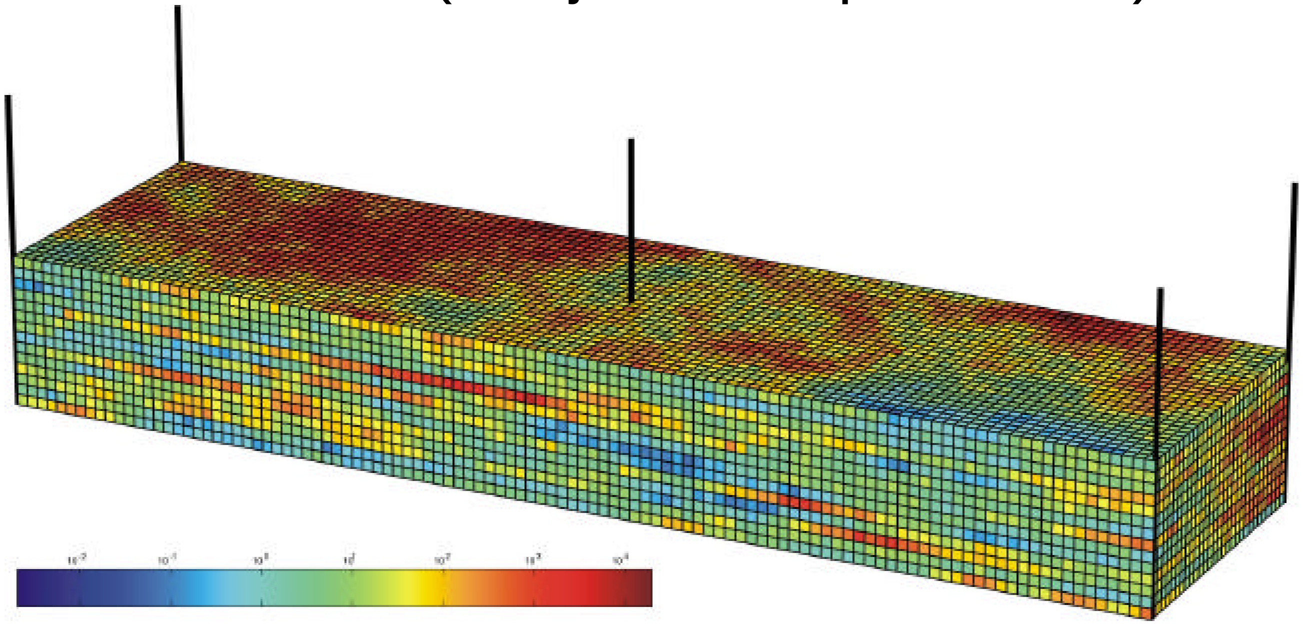
  - Compute streamlines for the velocity field $\mathbf{v}_T$
  - System of saturation equations (along each streamline)

$$\partial_t \begin{pmatrix} S_w \\ S_g \end{pmatrix} + \partial_t \begin{pmatrix} f_w \\ f_g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{where } t(s) \equiv \int_0^s \frac{1}{|v_T|} \mathrm{d}x$$

NUMERICAL SIMULATIONS

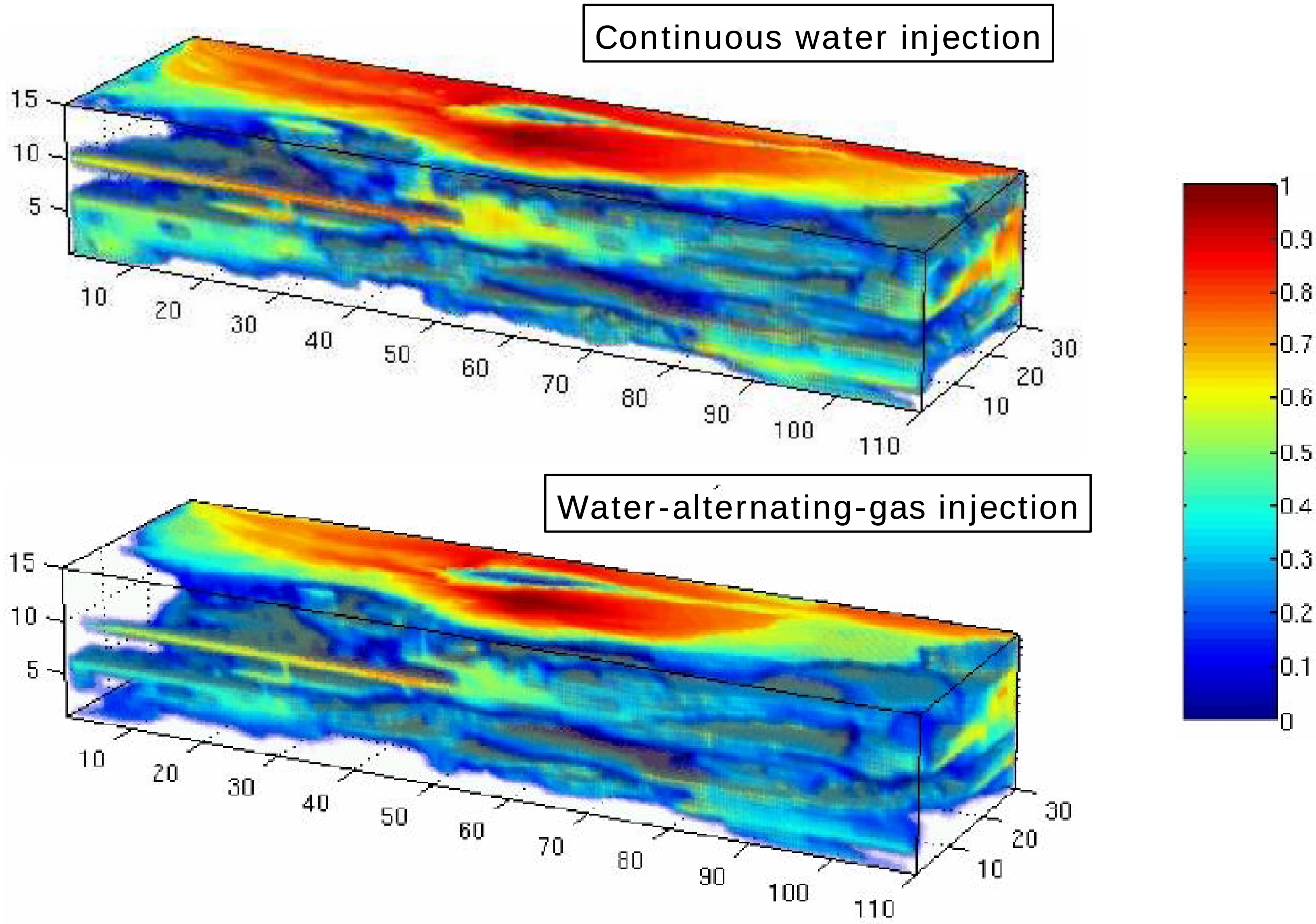
(Lie and J.: *CGEOS* submitted)

- Highly heterogeneous, shallow-marine formation, taken from the **SPE10 comparative solution project**
  - Permeability variations of 6 orders of magnitude
  - Five vertical wells (1 injector, 4 producers)



  - Two different injection schemes:
    - (1) Continuous water injection
    - (2) Water-alternating-gas injection (WAG)

WATER SATURATION AFTER 2000 DAYS

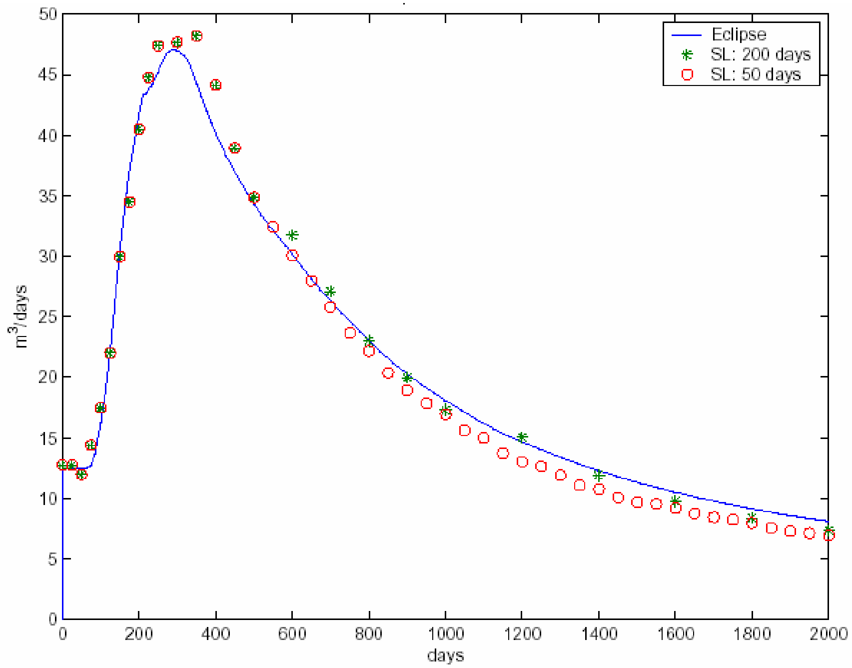


FLUID PRODUCTION

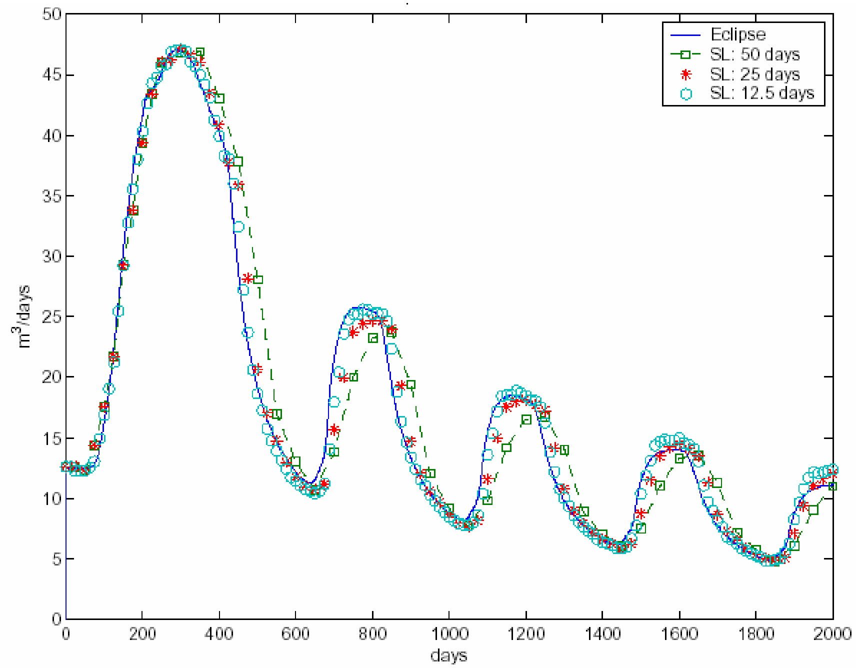
- Comparison of fluid recovery predictions against the commercial reservoir simulator Eclipse® (Schlumberger)

- **Oil production rate**

Continuous water injection

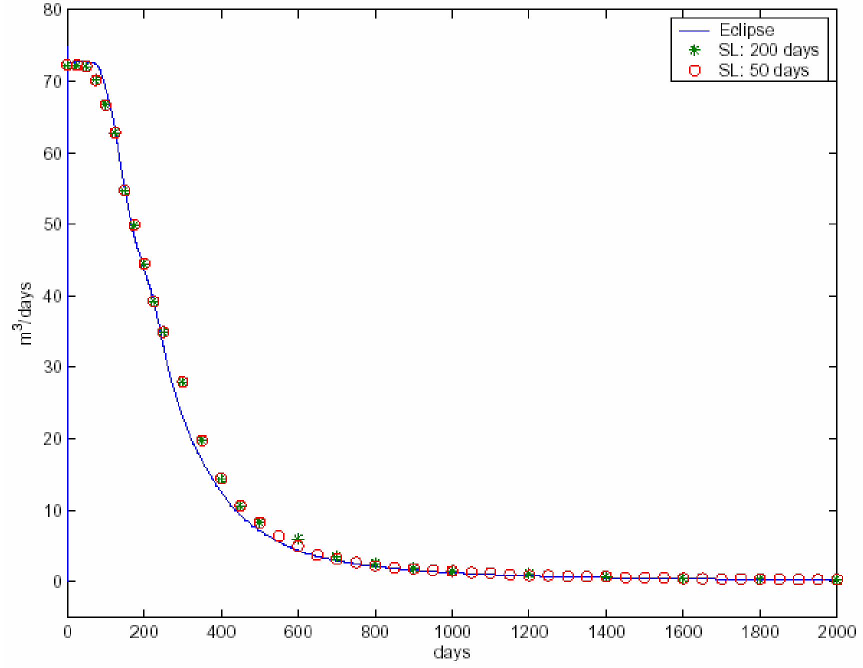


Water-alternating-gas injection

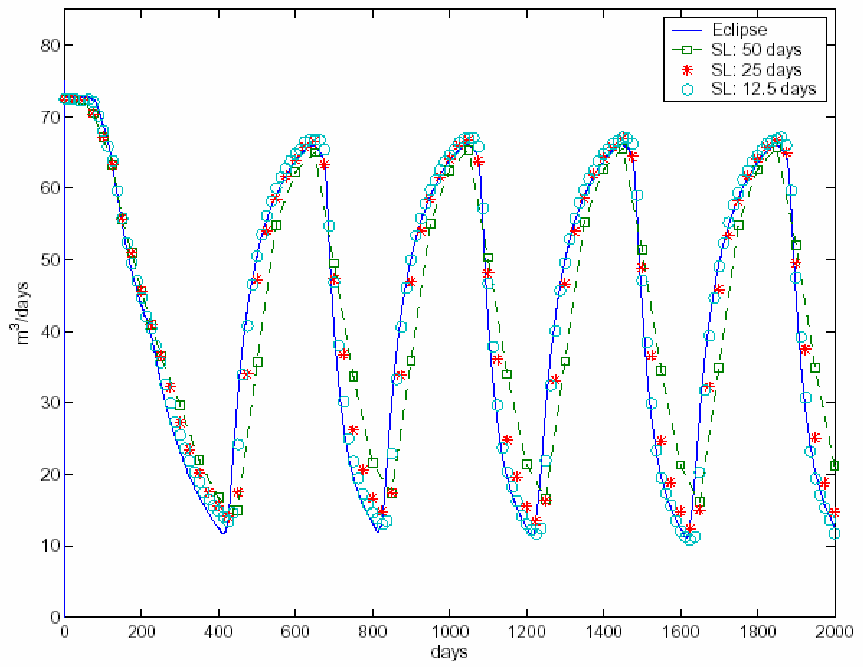


■ Gas production rate

Continuous water injection

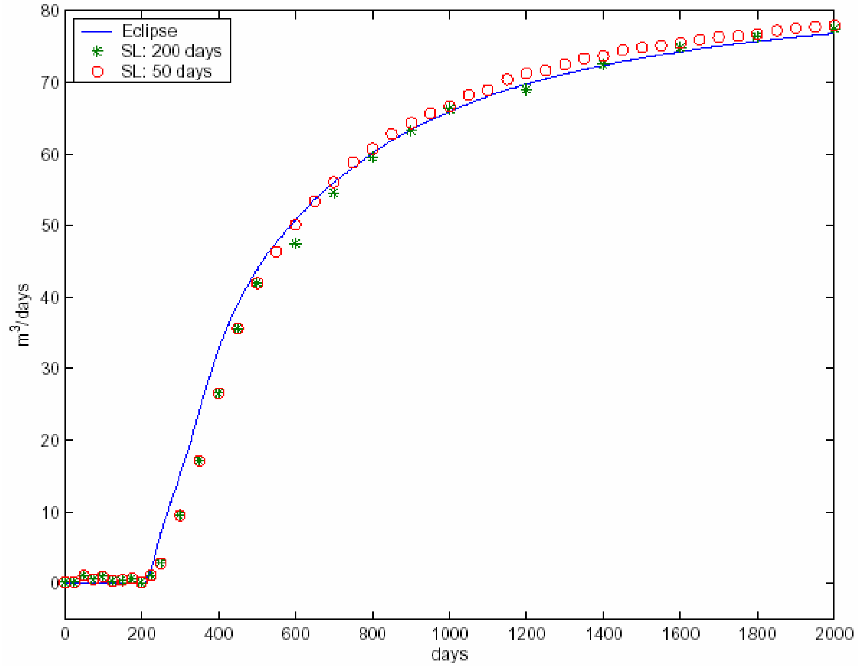


Water-alternating-gas injection

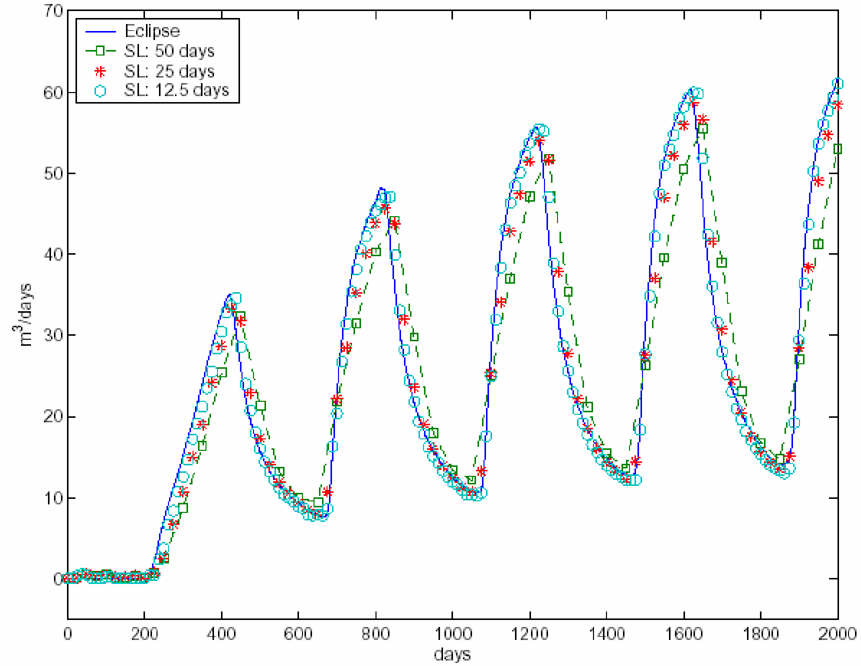


■ Water production rate

Continuous water injection



Water-alternating-gas injection



■ CPU times:

	Inyección de agua	Inyección alterna
ECLIPSE	1h 22min	8h 20min
Streamline	50min (**d***t* = 200 días)	2h 13min (**d***t* = 25 días)

CONCLUSIONS

- The integration of **analytical Riemann solvers**, the front-tracking method, and streamline simulation, offers the potential for fast and accurate prediction of three-phase flow in highly-heterogeneous reservoirs

FUTURE WORK

- **Extend the Riemann solver**
  - Residual saturaciones residuales
  - Relative permeability hysteresis
  - Fluid miscibility and compositional effects

- **Extend the streamline simulator**
  - Gravity, compressibility, and capillary pressure effects

PUBLICATIONS

- R. Juanes. *Displacement Theory and Multiscale Numerical Modeling of Three-Phase Flow.* PhD Dissertation, University of California, Berkeley, 2003.
- R. Juanes, T.W. Patzek. Relative permeabilities for strictly hyperbolic models of three-phase flow in porous media. *Transport in Porous Media* (accepted, in press).
- R. Juanes, T.W. Patzek. Analytical solution to the Riemann problem of three-phase flow in porous media. *Transport in Porous Media*, **55**(1):47-70, 2004.
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- R. Juanes, K.-A. Lie, V. Kippe. A front-tracking method for hyperbolic three-phase models. In *Proceedings* of ECMOR IX, Cannes, France, 2004.
- K.-A. Lie, R. Juanes. A front-tracking method for the simulation of three-phase flow in porous media. *Computational Geosciences* (in review).