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Low-Frequency Asymptotic Analysis of Seismic Reflection From a Fluid-Saturated Medium

D. B. SILIN[1,⋆], V. A. KORNEEV[1], G. M. GOLOSHUBIN[2] and T. W. PATZEK[3]
[1]*Lawrence Berkeley National Laboratory, 1 Cyclotron Road, MS 90-1116, Berkeley, CA 94720, U.S.A.*
[2]*Department of Geosciences, University of Houston, 504 Science and Research Bldg 1 Houston, TX 77204-5006, U.S.A.*
[3]*University of California, Berkeley, 437 Davis Hall, Berkeley, CA 94720, U.S.A.*

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Abstract. Reflection of a seismic wave from a plane interface between two elastic media does not depend on the frequency. If one of the media is poroelastic and fluid-saturated, then the reflection becomes frequency-dependent. This paper presents a low-frequency asymptotic formula for the reflection of seismic plane p-wave from a fluid-saturated porous medium. The obtained asymptotic scaling of the frequency-dependent component of the reflection coefficient shows that it is asymptotically proportional to the square root of the product of the reservoir fluid mobility and the frequency of the signal. The dependence of this scaling on the dynamic Darcy's law relaxation time is investigated as well. Derivation of the main equations of the theory of poroelasticity from the dynamic filtration theory reveals that this relaxation time is proportional to Biot's tortuosity parameter.

Key words: low-frequency signal, Darcy's law, seismic reflection.

1. Introduction

When a seismic wave interacts with a boundary between elastic and fluid-saturated media, some energy of the wave is reflected and the rest is transmitted or dissipated. It is well known that both the transmission and reflection coefficients from a fluid-saturated porous medium are functions of frequency (Geertsma and Smit, 1961; Dutta and Ode, 1983; Santos *et al.*, 1992; Denneman *et al.*, 2002). Recently, low-frequency signals were successfully used in obtaining high-resolution images of oil and gas reservoirs (Goloshubin and Bakulin, 1998; Goloshubin and Korneev, 2000; Castagna *et al.*, 2003) and in monitoring underground gas storage (Korneev *et al.*, 2004). Therefore, understanding the behavior of the

⋆Author for correpondence: e-mail: dsilin@lbl.gov

reflection coefficient at the low-frequency end of the seismic spectrum is of special importance.

The main objective of this paper is to obtain an asymptotic representation of the reflection of seismic signal from a fluid-saturated porous medium in the low-frequency domain. More specifically, we derive a simple formula, where the frequency-dependent component of the reflection coefficient is proportional to the square root of the product of frequency of the signal and the mobility of the fluid in the reservoir. This scaling can be different depending on the magnitude of the tortuosity factor. Since the latter is proportional to dynamic Darcy's law relaxation time, it can be evaluated from a flow experiment or using microscopic-scale flow modeling (Patzek, 2001).

We derive wave propagation equations from the basic principles of the theory of filtration. This is done, in particular, to verify that both the filtration and poroelasticity theories are based on a common foundation. We retain the equations needed in the asymptotic analysis that follows, skipping details where the calculations are similar to those in the classical works by Biot (1956a,b, 1962).

Fluid flow in an elastic porous medium is the subject of both filtration theory (Muskat, 1937; Polubarinova-Kochina, 1962; Bear, 1972; Barenblatt *et al.*, 1990) and the theory of poroelasticity (Frenkel, 1944; Gassmann, 1951; Biot, 1956a,b, 1962; Wang, 2000). The filtration theory usually assumes steady-state or transient processes where the macroscopic transition times are significantly longer than the transition times of the local microscopic processes. The poroelasticity theory includes a model of acoustic wave propagation in fluid-saturated elastic media, where the macroscopic transition times are short and, therefore, the concept of steady-state fluid flow may be inapplicable.

To obtain a system of equations characterizing fluid–solid interactions in a macroscopically homogeneous elastic fluid-saturated porous medium, we adopt *relaxation filtration* (Alishaev and Mirzadzhanzadeh, 1975; Molokovich *et al.*, 1980; Molokovich, 1987), which employs a *relaxation time* to account for the inertial and nonequilibrium effects in fluid flow, thus extending the classical Darcy's law (Darcy, 1856; Hubbert, 1940, 1956). Originally, Darcy's law was formulated for steady-state flow (Darcy, 1856). It is recognized that non-equilibrium effects are important in two-phase flow (Barenblatt, 1971; Barenblatt and Vinnichenko, 1980), (see also Silin and Patzek, 2004). However, due to local heterogeneities, they are also important in single-phase flow.

Further, it is demonstrated in Sections 2 and 3 that under different assumptions, the equations obtained here can be transformed either into Biot's wave equations (Biot, 1956a,b, 1962), or into the elastic pressure diffusion equation (Muskat, 1937; Matthews and Russell, 1967; Barenblatt *et al.*, 1990).

In the original Biot's works (1956a,b, 1962), the wave equations of poroelasticity were derived from the Hamiltonian least-action principle. In order to close the system, an introduction of a parameter having dimension of density was needed. This parameter is related to a dimensionless tortuosity factor characterizing the complexity of the pore space geometry in natural rocks. There are several definitions of tortuosity in the literature, (see, e.g., Bear, 1972). In Biot's derivation, the tortuosity factor statistically characterizes the heterogeneity of the local fluid velocity field (Biot, 1962). The way this tortuosity factor and the above-mentioned relaxation time enter the equations leads to the conclusion that they are linearly related to each other. The magnitude of the relaxation time and, hence, the value of the tortuosity, affects the way the reflection coefficient depends on frequency. Since the magnitude of the tortuosity in Biot's equations ranges, in general, between one and infinity (Molotkov, 1999), it is very important to know the tortuosity factors for different types of rock.

Over the last fifty years, a significant effort has been spent on the investigations of attenuation of Biot's waves, (see e.g., Pride and Berryman, 2003a,b and the references therein). It has been noticed that there must be a relation between the dependence of the attenuation on the wave frequency and the permeability of the reservoir (Pride *et al.*, 2003). In many cases, the attenuation coefficient can be obtained in an explicit, but quite cumbersome, form. Computation of the reflection coefficient is even more complex because it additionally requires inversion of a matrix. However, for a robust reservoir imaging procedure, a simple asymptotic expression is needed.

Low-frequency limit of Biot's theory was studied using homogenization technique (Auriault and Royer, 2002). In that work, the authors conclude that for a variety of media saturated with slightly compressible fluids, the distinction between Biot's (1956a) and Gassman's (1951) theories diminishes as the frequency tends to zero.

In this study, we obtain a simple asymptotic expression where the role of the reservoir fluid mobility is transparent. We focus on the simplest case of normal reflection of a p-wave.

In addition, we assume that rock grains are practically incompressible, so that all deformations of the rock and the pore space are due to the rearrangements of the grains. The scaling relationship obtained in Section 6 below has been successfully applied for imaging of oil reservoir productivity (Korneev *et al.*, 2004).

The layout of the paper is as follows. In Section 2, the main equations of the model are derived from the principles of filtration theory. In Section 3, the obtained relationships are compared with Biot's equations and the pressure diffusion model. In Section 4, we define a dimensionless small parameter for the asymptotic analysis of the known harmonic-wave solution to the equations of poroelasticity. In Section 5, the boundary conditions for the reflection problem are formulated. An asymptotic expression for the reflection coefficient with respect to the small parameters introduced in Section 4 is obtained in Section 6. In Section 7, we elaborate on how the relaxation time and tortuosity affect the asymptotic analysis.

2. Fluid-Solid Skeleton Interaction Equations

Consider a homogeneous porous medium M saturated with a viscous fluid. The grains of the solid skeleton are displaced by an elastic wave. It is assumed that a plane p-wave is propagating along the x-axis of a fixed Cartesian coordinate system. Thus, after averaging over a plane orthogonal to x, the only nonzero component of the displacement is the x-component, and the mean displacement is one-dimensional. Due to the skeleton deformation, the grains are rearranged. We assume that the rearrangement occurs through elastic deformations of the cement bonds between the grains. Such an assumption is natural in many situations considered in hydrology and is quite common in the geophysical literature as well, (see, e.g., Denneman *et al.*, 2002).

In general, deformations result in energy dissipation. In this paper, for simplicity, it is assumed that these energy losses are much smaller than the losses through viscous friction in flow of the reservoir fluid. Further, we assume that the rock is 'clean', so that the total mass and volume of the bonds are small relative to those of the grains. Thus, for the bulk density of the "dry" skeleton ϱ we have

$$\varrho = (1-\phi)\varrho_{\mathrm{g}}, \tag{1}$$

where ϱ_{g} is the density of the grains and ϕ is the porosity. If we neglect the microscopic rotational motions of the grains, the mean density of momentum of the drained skeleton is given by

$$\varrho\frac{\partial u}{\partial t} = (1-\phi)\varrho_{\mathrm{g}}\frac{\partial u}{\partial t}, \tag{2}$$

where u is the mean displacement of the skeleton grains in the x-direction and t denotes time.

The skeleton deformations change the stress field. We consider only small variations of parameters near a reference configuration, where all forces are at equilibrium. It is natural to assume that the shear stresses are uniformly distributed over directions orthogonal to x. In general, even uniformly distributed shear stress influences the rearrangement of the skeleton. However, the assumption of stiff grains and small-volume bonds allows us to neglect this influence. The x-component, σ_x, of the stress implied by a displacement of the solid skeleton, u, at a constant fluid pressure, that is similar to effective stress (Terzaghi and Peck, 1948), can be measured by the elastic forces acting on a unit (bulk) area in a plane orthogonal to x. Linear elasticity hypothesis suggests that for small displacements, the stress σ_x and the displacement u are linearly related:

$$\sigma_x = \frac{1}{\beta}\frac{\partial u}{\partial x}. \tag{3}$$

Here $\beta = 1/K$ is the drained bulk compressibility, or the inverse of the bulk modulus K. We retain the subscript x in Equation (3) just to emphasize that here we focus on a one-dimensional case only.

The motion of the reservoir fluid can be characterized by the superficial or Darcy velocity W measured relative to the skeleton. This means, that if we imagine a small surface element moving along with the local displacement of the grains, then the volumetric fluid flux through this surface is equal to the projection of W on the unit normal vector to the surface. The average velocity v_{f} of the fluid particles relative to the skeleton is related to the Darcy velocity by equation

$$\phi v_{\mathrm{f}} = W. \tag{4}$$

The total fluid pressure-related force acting on the solid skeleton is equal to $-(\partial p/\partial x)$ (Polubarinova-Kochina, 1962; Wang, 2000). A small volume of the medium, δV, contains $\varrho\delta V$ mass of rock material and $\phi\varrho_{\mathrm{f}}\delta V$ mass of fluid. Here ϱ_{f} is the density of the fluid. Hence, the momentum of moving fluid per unit bulk volume is

$$\phi\varrho_{\mathrm{f}}\left(\frac{\partial u}{\partial t} + v_{\mathrm{f}}\right) = \phi\varrho_{\mathrm{f}}\frac{\partial u}{\partial t} + \varrho_{\mathrm{f}} W. \tag{5}$$

Thus, the momentum balance per unit bulk volume is

$$\varrho_{\mathrm{b}}\frac{\partial^2 u}{\partial t^2} + \varrho_{\mathrm{f}}\frac{\partial W}{\partial t} = \frac{1}{\beta}\frac{\partial^2 u}{\partial x^2} - \frac{\partial p}{\partial x}, \tag{6}$$

where ϱ_{b} is the bulk density of the fluid-saturated medium

$$\varrho_{\mathrm{b}} = (1-\phi)\varrho_{\mathrm{g}} + \phi\varrho_{\mathrm{f}} = \varrho + \phi\varrho_{\mathrm{f}}. \tag{7}$$

Now, consider the motion of the fluid. According to Darcy's law, at steady-state conditions

$$W = -\varrho_{\mathrm{f}}\frac{\kappa}{\eta}\frac{\partial \Phi}{\partial x}, \tag{8}$$

where κ is the permeability of the medium, η is the viscosity of the fluid and Φ is the flow potential (Hubbert, 1940, 1956). We consider only small perturbations near an equilibrium configuration and the Darcy velocity W is measured relative to the porous medium. Hence, the differential of potential Φ is amended with a term characterizing additional pressure drop due to the accelerated motion of the skeleton

$$\mathrm{d}\Phi = \frac{\mathrm{d}p}{\varrho_\mathrm{f}} + \frac{\partial^2 u}{\partial t^2}\,\mathrm{d}x. \tag{9}$$

Darcy's law (8) is for steady-state flow. If flow is transient, for example, due to abrupt changes in the pressure field, Equation (8) may need to be modified in order to account for the inertial and relaxation effects. To derive the respective equation, we use an argument similar to that in Barenblatt and Vinnichenko (1980). As the pressure gradient changes, the local redistribution of the pressure field does not occur instantaneously because it includes microscopic fluid flow along and between the pores. Thus, the gradient of flow potential determines some combination of Darcy velocity and 'Darcy acceleration'

$$\Psi\left(W, \tau\frac{\partial W}{\partial t}\right) = -\varrho_\mathrm{f}\frac{\kappa}{\eta}\frac{\partial\Phi}{\partial x}. \tag{10}$$

Clearly, $\Psi(W, 0) = W$. At low-frequency limit, the acceleration component is small, hence a linearization with respect to the second parameter yields

$$W + \tau\frac{\partial W}{\partial t} = -\varrho_\mathrm{f}\frac{\kappa}{\eta}\frac{\partial\Phi}{\partial x}. \tag{11}$$

Here τ is a characteristic redistribution time.

Such a modification of Darcy's law was proposed by Alishaev (1974), Alishaev and Mirzadzhanzadeh (1975) using different assumptions. In multiphase flow, similar considerations were used to model nonequilibrium effects at the front of water–oil displacement and spontaneous imbibition (Barenblatt, 1971; Barenblatt and Vinnichenko, 1980). Some estimates of the relaxation time, based on an interpretation of experiments, were reported in Molokovich *et al.* (1980), Molokovich (1987), and Dinariev and Nikolaev (1990). Apparently, the relaxation time is a function of the pore space geometry, fluid viscosity η, and compressibility β_f. Dimensional analysis then suggests that $\tau = \eta\beta_\mathrm{f}F(\kappa/L^2)$, where L is the characteristic size of an elementary representative volume of the medium, and F is some dimensionless function. Time τ is apparently related to the tortuosity factor (Biot, 1962). This relationship is discussed in more detail below.

Summing up, we arrive at the following equation characterizing the dynamics of fluid flow

$$W + \tau\frac{\partial W}{\partial t} = -\frac{\kappa}{\eta}\frac{\partial p}{\partial x} - \varrho_\mathrm{f}\frac{\kappa}{\eta}\frac{\partial^2 u}{\partial t^2}. \tag{12}$$

The assumption that both skeleton displacement u and Darcy velocity W are just small perturbations near some equilibrium values is also applied to the fluid pressure p. Only these small variations have non-zero derivatives. Therefore, we retain only the terms, which are linear with respect to small perturbations. A system of momentum balance equations accounting for convective momentum transport in terms of microscopic fluid velocities is presented in Nikolaevskii (1996). In Equations (6) and (12), Darcy velocity is used in conjunction with dynamic version of Darcy's low.

The mass balances for the fluid and the solid skeleton are

$$\frac{\partial(\varrho_{\mathrm{f}}\phi)}{\partial t}=-\frac{\partial\left(\varrho_{\mathrm{f}}W+\phi\varrho_{\mathrm{f}}\frac{\partial u}{\partial t}\right)}{\partial x}, \tag{13}$$

$$\frac{\partial\varrho}{\partial t}=-\frac{\partial}{\partial x}\left(\varrho\frac{\partial u}{\partial t}\right). \tag{14}$$

For the fluid, we apply the isothermal compressibility law (Landau and Lifschitz, 1959), that is, for small fluid pressure perturbation

$$\frac{\mathrm{d}\varrho_{\mathrm{f}}}{\varrho_{\mathrm{f}}}=\beta_{\mathrm{f}}\,\mathrm{d}p. \tag{15}$$

Hence, Equation (13) can be rewritten as

$$\frac{\partial\phi}{\partial t}+\phi\beta_{\mathrm{f}}\frac{\partial p}{\partial t}=-\frac{\partial W}{\partial x}-\phi\frac{\partial^2 u}{\partial x\partial t}-W\frac{\partial\varrho_{\mathrm{f}}}{\partial x}-\frac{1}{\varrho_{\mathrm{f}}}\frac{\partial}{\partial x}(\phi\varrho_{\mathrm{f}})\frac{\partial u}{\partial t}. \tag{16}$$

Since the parameter variations are small, and only the perturbed components have nonzero derivatives, the last two terms in Equation (16) are of higher order and can be neglected.

With $\rho=(1-\phi)\rho_{\mathrm{g}}$, Equation (14) takes on the form

$$-\frac{\partial\phi}{\partial t}+(1-\phi)\frac{1}{\varrho_{\mathrm{g}}}\frac{\partial\varrho_{\mathrm{g}}}{\partial t}=-\frac{1}{\varrho_{\mathrm{g}}}(1-\phi)\frac{\partial\varrho_{\mathrm{g}}}{\partial x}\frac{\partial u}{\partial t}+\frac{\partial\phi}{\partial x}\frac{\partial u}{\partial t}-(1-\phi)\frac{\partial^2 u}{\partial x\,\partial t}. \tag{17}$$

The smallness of perturbations implies that the first two terms on the right-hand side of the last equation can be dropped. Further on, the perturbation of grain density is a linear function of the perturbations of stress and fluid pressure, that is

$$\frac{1}{\varrho_{\mathrm{g}}}\mathrm{d}\varrho_{\mathrm{g}}=\beta_{\mathrm{gs}}\mathrm{d}\sigma_x+\beta_{\mathrm{gf}}\,\mathrm{d}p, \tag{18}$$

where β_{gs} and β_{gf} are the respective compressibility coefficients. Thus, Equation (17) can be written as

$$\frac{\partial\phi}{\partial t}=(1-\phi)\beta_{\mathrm{gf}}\frac{\partial p}{\partial t}+(1-\phi)\left(1+\frac{\beta_{\mathrm{gs}}}{\beta}\right)\frac{\partial^2 u}{\partial x\,\partial t}. \tag{19}$$

A combination of this last result with Equation (16) leads to the following relationship

$$\left(1+(1-\phi)\frac{\beta_{\mathrm{gs}}}{\beta}\right)\frac{\partial^2 u}{\partial x\,\partial t}+(\phi\beta_{\mathrm{f}}+(1-\phi)\beta_{\mathrm{gf}})\frac{\partial p}{\partial t}=-\frac{\partial W}{\partial x}. \tag{20}$$

The grain compressibility is much smaller than the compressibility of the fluid or the skeleton:

$$\beta_{\mathrm{gf}}\ll\beta_{\mathrm{f}} \quad \text{and} \quad \beta_{\mathrm{gs}}\ll\beta. \tag{21}$$

This means that bulk deformation occurs only through the porosity perturbations. Thus, Equation (20) can be further reduced to

$$\frac{\partial^2 u}{\partial x\partial t}+\phi\beta_{\mathrm{f}}\frac{\partial p}{\partial t}=-\frac{\partial W}{\partial x}. \tag{22}$$

Equation (22) states that the amount of fluid volume packed into a unit bulk volume per unit time is equal to minus the divergence of the absolute fluid velocity. This fluid redistribution occurs due to fluid compression and porosity variation. Note that Equations (20) and (22) are mathematically similar. Below, we use the more general mass balance equation (20) unless it exceedingly complicates the calculations.

To summarize, we have obtained a closed system of three equations (6), (12), and (20) with three unknown functions of t and x: the skeleton displacement u, the fluid pressure p, and the Darcy velocity W.

3. Relationship to Biot's Poroelasticity and Pressure Diffusion Equations

In this section, we demonstrate that under the assumptions formulated in Section 2 Equations (6), (12), and (20) can be reduced to the system of equations obtained by Biot (1956a, 1962), (see also Dutta and Ode, 1979). At the same time, neglecting the inertial terms in these equations, leads to the pressure diffusion equation used in hydrology and petroleum engineering for well test analysis (see Theis, 1935; Jacob, 1940 or the books Matthews and Russell, 1967; Barenblatt *et al.*, 1990).

To recover Biot's poroelasticity equations, the assumption of grain incompressibility, Equations (21), is applied. For small oscillatory deformations of the skeleton and fluctuations of the fluid flow, a 'superficial' displacement w of the fluid relative to the skeleton can be introduced, so that

$$W=\frac{\partial w}{\partial t}. \tag{23}$$

Note that inasmuch as w is related by Equation (23) to the Darcy velocity of the fluid, it is different from the average microscopic fluid displacement. Substitution of (23) into Equation (22) yields

$$\frac{\partial^2 u}{\partial x\,\partial t}+\phi\beta_{\rm f}\frac{\partial p}{\partial t}=-\frac{\partial^2 w}{\partial t\,\partial x}. \tag{24}$$

By integration in t and differentiation in x, we obtain

$$\frac{\partial p}{\partial x}=-\frac{1}{\phi\beta_{\rm f}}\frac{\partial^2 u}{\partial x^2}-\frac{1}{\phi\beta_{\rm f}}\frac{\partial^2 w}{\partial x^2}. \tag{25}$$

In this derivation, we have used the assumption of the smallness of the rock–fluid system oscillations near an equilibrium configuration. Otherwise, due to the integration, Equation (25) should include an unknown function of x. Substitution of Equation (23) and the result (25) in Equations (6) and (12) yields:

$$\varrho_{\rm b}\frac{\partial^2 u}{\partial t^2}+\varrho_{\rm f}\frac{\partial^2 w}{\partial t^2}=\left(\frac{1}{\beta}+\frac{1}{\phi\beta_{\rm f}}\right)\frac{\partial^2 u}{\partial x^2}+\frac{1}{\phi\beta_{\rm f}}\frac{\partial^2 w}{\partial x^2}, \tag{26}$$

$$\varrho_{\rm f}\frac{\partial^2 u}{\partial t^2}+\tau\frac{\eta}{\kappa}\frac{\partial^2 w}{\partial t^2}=\frac{1}{\phi\beta_{\rm f}}\frac{\partial^2 u}{\partial x^2}+\frac{1}{\phi\beta_{\rm f}}\frac{\partial^2 w}{\partial x^2}-\frac{\eta}{\kappa}\frac{\partial w}{\partial t}. \tag{27}$$

Under the assumptions formulated above, Equations (26) and (27) are equivalent to the Biot system of equations (8.34) (Biot, 1962):

$$\frac{\partial^2}{\partial t^2}(\varrho_{\rm b}u+\varrho_{\rm f}w)=\frac{\partial}{\partial x}\left(A_{11}\frac{\partial u}{\partial x}+M_{11}\frac{\partial w}{\partial x}\right),$$

$$\frac{\partial^2}{\partial t^2}(\varrho_{\rm f}u+mw)=\frac{\partial}{\partial x}\left(M_{11}\frac{\partial u}{\partial x}+M\frac{\partial w}{\partial x}\right)-\frac{\eta}{\kappa}\frac{\partial w}{\partial t}.$$

Comparing the individual terms, we can establish a relationship between the relaxation time and the tortuosity factor. Namely, the relaxation time τ is related to the dynamic coupling coefficient m (Biot, 1962) through the inverse mobility ratio η/κ. The dynamic coupling coefficient is often expressed through the tortuosity factor T: $m=T\varrho_{\rm f}/\phi$. Hence, for the tortuosity and relaxation time, we obtain the following relationship:

$$T=\tau\frac{\eta\phi}{\kappa\varrho_{\rm f}} \quad \text{or} \quad \tau=T\frac{\kappa\varrho_{\rm f}}{\eta\phi}. \tag{28}$$

Comparison of the elastic coefficients reveals that under the assumption of isotropic porous medium and incompressible grains (the Biot–Willis coefficient $\alpha=K/H\approx 1$, and $K_{\rm u}=K+K_{\rm f}/\phi$), the Biot coefficients are constant and equal to

$$A_{11}=K_{\rm u}\approx\frac{1}{\beta}+\frac{1}{\phi\beta_{\rm f}} \quad \text{and} \quad M_{11}=M=K_{\rm u}B\approx\frac{1}{\phi\beta_{\rm f}}, \tag{29}$$

where $K_{\rm u}$ is the undrained bulk modulus, and $B=R/H$ is Skempton's coefficient, $1/H$ being the poroelastic expansion coefficient, and $1/R$ the unconstrained specific storage coefficient.

For derivation of the pressure diffusion equation, we assume that the characteristic time t_D of the process is large in comparison with the relaxation time τ and the displacements of the skeleton are much smaller then the characteristic length scale of the process L:

$$t_D \gg \tau \quad \text{and} \quad u \ll L. \tag{30}$$

Under this assumption, the second-order time derivatives of displacement u and time derivatives of Darcy velocity W in Equations (6) and (12) can be dropped:

$$\frac{\partial p}{\partial x} = \frac{1}{\beta}\frac{\partial^2 u}{\partial x^2}, \tag{31}$$

$$W = -\frac{\kappa}{\eta}\frac{\partial p}{\partial x}. \tag{32}$$

By integrating Equation (31) in x and differentiating in t, we obtain

$$\frac{\partial^2 u}{\partial t \partial x} = \beta \frac{\partial p}{\partial t}. \tag{33}$$

Formally, the integration with respect to x is defined up to a function of time, which is constant due to the constant pressure boundary condition at infinity. This constant is later cancelled by the differentiation with respect to t. Finally, by a substitution of Equations (32) and (33) into (22), we obtain

$$\phi\left(\frac{\beta}{\phi + \beta_f}\right)\frac{\partial p}{\partial t} = \frac{\kappa}{\eta}\frac{\partial^2 p}{\partial x^2}. \tag{34}$$

This last equation is the pressure diffusion equation routinely used in well test analysis (Matthews and Russell, 1967; Barenblatt *et al.*, 1990).

4. Plane Compressional Wave: An Asymptotic Solution

In this Section, we obtain the low-frequency asymptotic expressions for p-waves in fluid-saturated poroelastic media. These results are used in Section 6 in asymptotic analysis of the reflection coefficient.

To transform the system of Equations (6), (12), and (20) obtained in Section 2, we introduce the dimensionless pressure

$$P = \phi\beta_f p \tag{35}$$

and the hydraulic diffusivity

$$D = \frac{\kappa}{\phi \beta_f \eta}. \tag{36}$$

Dividing Equation 6 by ϱ_b and putting

$$v_b^2 = \frac{1}{\beta \varrho_b} \quad \text{and} \quad v_f^2 = \frac{1}{\phi \beta_f \varrho_b}, \tag{37}$$

we obtain

$$\frac{\partial^2 u}{\partial t^2} + \frac{\varrho_f}{\varrho_f} \frac{\partial W}{\partial t} = v_b^2 \frac{\partial^2 u}{\partial x^2} - v_f^2 \frac{\partial P}{\partial x}, \tag{38}$$

$$\lambda_f \frac{\partial^2 u}{\partial t^2} + W + \tau \frac{\partial W}{\partial t} = -D \frac{\partial P}{\partial x}, \tag{39}$$

$$\gamma_1 \frac{\partial^2 u}{\partial x \, \partial t} + \gamma_2 \frac{\partial P}{\partial t} = -\frac{\partial W}{\partial x}, \tag{40}$$

where

$$\lambda_f = \varrho_f \frac{\kappa}{\eta} \tag{41}$$

is the ‘kinematic’ mobility of the fluid, and

$$\gamma_1 = 1 + (1 - \phi) \frac{\beta_{gs}}{\beta} \quad \text{and} \quad \gamma_2 = 1 + (1 - \phi) \frac{\beta_{gf}}{\phi \beta_f}. \tag{42}$$

Coefficient λ_f has the dimension of time. Assumption (21) imply that both dimensionless coefficients γ_1 and γ_2 are close to one. The system of Equations (38)–(40) is similar to Biot’s system; however, it uses fluid pressure and Darcy velocity, that are more typical of filtration theory. System (38)–(40) admits a solution, which is the sum of slow and fast waves (Biot, 1956a). Asymptotic analysis of these waves is our next goal.

A plane-wave solution to Equations (38)–(40) has the form

$$u = U_s e^{i(\omega t - kx)}, \qquad W = W_f e^{i(\omega t - kx)}, \qquad P = P_0 e^{i(\omega t - kx)}. \tag{43}$$

Substitution of Equation (43) into (38)–(40) and some algebra yield

$$W_f = -i\omega \gamma_1 U_s + \omega \gamma_2 \frac{P_0}{k} \tag{44}$$

or

$$W_f = i\omega(-\gamma_1 + \gamma_2 \xi) U_s = v \left(-\frac{\gamma_1}{\xi} + \gamma_2 \right) P_0, \tag{45}$$

where

$$v = \frac{\omega}{k} \quad \text{and} \quad \xi = -\frac{i P_0}{k U_s} \tag{46}$$

Denote

$$\tau_D = \frac{D}{v_f^2} = \frac{\kappa \varrho_b}{\eta}, \qquad \gamma_v = \frac{v_b^2}{v_f^2} = \frac{\phi \beta_f}{\beta} \quad \text{and} \quad \gamma_\varrho = \frac{\varrho_f}{\varrho_b} \tag{47}$$

The parameters γ_v and γ_ϱ are dimensionless. Taking into account Equation (41)

$$\lambda_f = \gamma_\varrho \tau_D. \tag{48}$$

The dimensionless relaxation time θ and dimensionless angular frequency ε are defined as

$$\theta = \frac{\tau}{\tau_D} \quad \text{and} \quad \varepsilon = \tau_D \omega. \tag{49}$$

Using these definitions, we obtain the following quadratic equation with respect to ξ:

$$\begin{aligned} &\left(\gamma_2 + i\varepsilon\left(-\gamma_2\gamma_\varrho + \theta\gamma_2\right)\right)\xi^2 + \\ &\quad + \left(-\gamma_1 + \gamma_2\gamma_v + i\varepsilon\left[-1 + \gamma_1\gamma_\varrho + (\gamma_\varrho - \theta\gamma_1) + \theta\gamma_2\gamma_v\right]\right)\xi + \\ &\quad + \left(-\gamma_1\gamma_v + i\varepsilon\gamma_v(\gamma_\varrho - \tau\gamma_1)\right) = 0. \end{aligned} \tag{50}$$

If we assume the permeability $\kappa \sim 1$ Darcy, that is $\kappa \sim 10^{-12}$ m^2, the viscosity of the fluid $\eta \sim 1$ cP $= 10^{-3}$ Pa-s, and the bulk density of the rock $\varrho_b \sim 10^3$ kg/m^3, then $\tau_D \sim 10^{-6}$ and $\varepsilon \preceq 10^{-3}$ for frequencies ω not exceeding $\sim$1 kHz. Since γ_1 and γ_2 are of the order of unity, ε (more accurately, $i\varepsilon$) is a small parameter in Equation (50). At $\varepsilon = 0$, there are two *real* roots

$$\xi_0^{(1)} = \frac{\gamma_1}{\gamma_2} \quad \text{and} \quad \xi_0^{(2)} = -\gamma_v. \tag{51}$$

By virtue of Equations (21) and (42), the absolute value of the first root ξ_0^1 is close to unity, whereas the absolute value of the second one is equal to $\phi\beta_f/\beta$, that is usually larger than one. We obtain two real asymptotic values for the complex velocity v

$$v_0^{(1)} = 0 \quad \text{and} \quad v_0^{(2)} = v_f \sqrt{\gamma_v + \frac{\gamma_1}{\gamma_2}}. \tag{52}$$

The first solution corresponds to the slow wave, whereas the second one is related to the fast wave.

The exact solution to Equation (50) is cumbersome and nontransparent. Therefore, we obtain an asymptotic solution directly from Equation (50) in the form

$$\xi = \xi_0 + \xi_1 i\varepsilon - \xi_2\varepsilon^2 \ldots \tag{53}$$

Using the notations

$$\begin{aligned} A_0 &= \gamma_2, & A_1 &= -\gamma_2\gamma_\varrho + \theta\gamma_2, \\ B_0 &= \gamma_2\gamma_v - \gamma_1, & B_1 &= -1 + \gamma_\varrho(1+\gamma_1) + \theta(\gamma_2\gamma_v - \gamma_1), \\ C_0 &= -\gamma_1\gamma_v, & C_1 &= \gamma_v(\gamma_\varrho - \theta\gamma_1), \end{aligned} \tag{54}$$

we obtain

$$\xi_1 = -\frac{A_1\xi_0^2 + B_1\xi_0 + C_1}{2A_0\xi_0 + B_0}. \tag{55}$$

Thus, the solutions corresponding to the slow and fast waves have, respectively, the following forms

$$\xi_1^{(1)} = \gamma_v \frac{1 - \gamma_\varrho(\gamma_2\gamma_v + \gamma_1)}{\gamma_1 + \gamma_2\gamma_v} \tag{56}$$

and

$$\xi_1^{(2)} = \frac{1}{\gamma_2}\frac{\gamma_1 - \gamma_\varrho(\gamma_2\gamma_v + \gamma_1)}{\gamma_1 + \gamma_2\gamma_v}. \tag{57}$$

Note, that since both $\gamma_1 \approx 1$ and $\gamma_2 \approx 1$, Equations (56) and (57) can be simplified

$$\xi_1^{(1)} = \gamma_v \frac{1 - \gamma_\varrho\gamma_v - \gamma_\varrho}{1 + \gamma_v}, \tag{58}$$

$$\xi_1^{(2)} = \frac{1}{\gamma_2}\frac{\gamma_1 - \gamma_\varrho\gamma_v - \gamma_\varrho}{1 + \gamma_v}. \tag{59}$$

In particular, $\xi_1^{(1)}$ and $\xi_1^{(2)}$ are independent of the permeability of the formation and the viscosity of the fluid. Note that the relaxation time also disappears from the first-order approximation of ξ for both the slow and fast wave. The latter circumstance is discussed in Section 7 below.

We further obtain that

$$v^{(1)} = \pm v_{\mathrm{b}}\sqrt{\frac{i\varepsilon}{\gamma_1 + \gamma_2\gamma_v} + \cdots} \tag{60}$$

and

$$v^{(2)} = \pm v_{\mathrm{f}}\sqrt{\gamma_v + \frac{\gamma_1}{\gamma_2}} + v_{\mathrm{f}} V_1 i\varepsilon + \cdots, \tag{61}$$

where V_1 is the first coefficient of the expansion of V in the powers of $i\varepsilon$. The last two equations, in a combination with equation (56), imply that

$$k^{(1)} = \pm\frac{1}{\tau_D v_b}\sqrt{\gamma_1+\gamma_2\gamma_v}\sqrt{-i\varepsilon}+\cdots, \tag{62}$$

$$k^{(2)} = \pm\frac{1}{\tau_D v_f}\frac{1}{\sqrt{\gamma_v+\dfrac{\gamma_1}{\gamma_2}}}\varepsilon+\cdots \tag{63}$$

The imaginary part of k must be negative. Therefore, from (62), we infer that

$$k^{(1)} = \frac{1}{\tau_D v_b}\sqrt{\gamma_1+\gamma_2\gamma_v}\,\frac{1-i}{\sqrt{2}}\sqrt{\varepsilon}+\cdots \tag{64}$$

and, respectively,

$$v^{(1)} = v_b\sqrt{\frac{1}{\gamma_1+\gamma_2\gamma_v}}\,\frac{1+i}{\sqrt{2}}\sqrt{\varepsilon}+\cdots \tag{65}$$

By virtue of Equations (51) and (45)

$$W_f = -i\omega(\gamma_1-\gamma_2\xi)U_s. \tag{66}$$

Furthermore, using Equations (53), we get for the fast wave

$$W_f^{\text{fast}} = -\varepsilon\omega\gamma_2\xi_1^{(2)}U_s^{\text{fast}}+\cdots \tag{67}$$

The right-hand side of the last equation is first-order small with respect to ε. In other words, at low-frequencies, the fast wave is almost a coherent oscillation of the skeleton and the fluid. At the same time, for the slow wave, the Darcy velocity amplitude is comparable with the amplitude of the time-derivative of the displacement

$$W_f^{\text{slow}} = -i\omega\,(\gamma_1+\gamma_2\gamma_v)\,U_s^{\text{slow}}+\cdots \tag{68}$$

5. Reflection: Boundary Conditions

Consider a normal incidence of a compressional elastic wave upon a plane interface $x=0$ separating media M_1 and M_2 occupying half-spaces $x<0$ and $x>0$, respectively, see Figure 1. Medium M_1 is ideal elastic solid, whereas medium M_2 is poroelastic fluid-saturated medium. The elastic

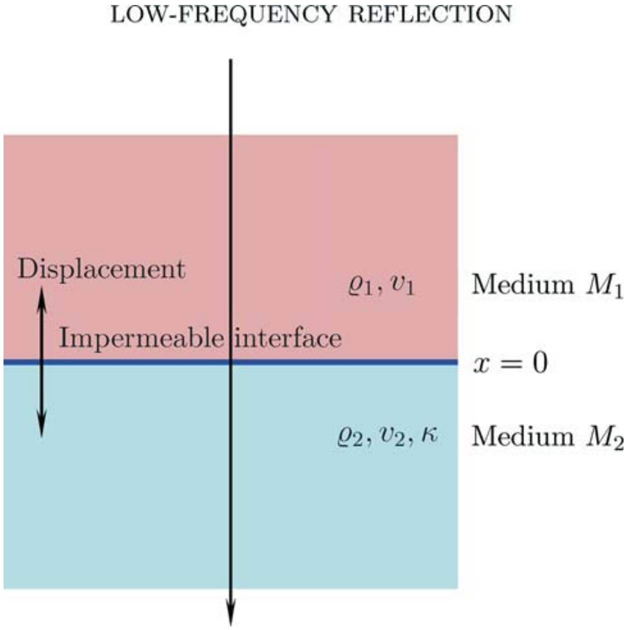

Figure 1. One-dimensional propagation of a low-frequency disturbance perpendicular to the impermeable interface between medium M_1 and porous, permeable solid M_2 fully saturated with a liquid.

properties of M_1 and solid skeleton of M_2 are characterized by the bulk densities ϱ_i and the speeds of sound v_i, $i = 1, 2$. We assume that the permeability of medium M_2 is κ and the boundary between the media is impermeable to fluid flow. To calculate the reflection coefficient, boundary conditions at the interface between the media, i.e., at $x = 0$, must be formulated.

Under the assumptions of Section 3, and neglecting the heterogeneities of the materials, we can assume that the displacements of the solid particles composing the media are parallel to x, and so is the flux of the fluid in the pore space. There is an important difference between the fluid and solid motion. The solid particles move more or less coherently near the respective equilibrium positions, whereas fluid particles move in a much more dispersed manner caused by the complexity of the pore space geometry. Only the mean volumetric flux or Darcy velocity of the moving fluid is parallel to x. This quantity is the result of averaging the microscopic fluid velocity field over a representative volume. In the case under consideration, such an averaging can be performed over a plane $x = \text{Const} > 0$.

Denote by u_1 and u_2 the displacements of the solid particles in media M_1 and M_2, respectively.

First, the continuity of the displacements and microscopic stresses requires that

$$u_1|_{x=0} = u_2|_{x=0}\,, \tag{69}$$

$$-\frac{1}{\beta_1}\frac{\partial u_1}{\partial x}\bigg|_{x=0} = -\frac{1}{\beta_2}\frac{\partial u_2}{\partial x}\bigg|_{x=0} + \phi p|_{x=0}\,. \tag{70}$$

Here we use the fact that the area of the contact between medium M_1 and the fluid saturating medium M_2 is a part of the total area proportional to the porosity of medium M_2.

Zero fluid flux through the boundary implies

$$W_{\mathrm{f}}|_{x=0} = 0. \tag{71}$$

Boundary conditions (69)–(71) will be used in the next section for investigation of the reflection coefficient.

6. Reflection Coefficient

To calculate the reflection coefficient, we substitute in boundary conditions (69)–(71) the sum of incident and reflected displacements in medium M_1

$$u_1 = U_1 \mathrm{e}^{i(\omega t - k_1 x)} + RU_1 \mathrm{e}^{i(\omega t + k_1 x)} \tag{72}$$

and the sum of slow and fast waves transmitted into medium M_2 expressed in terms of the fluid pressure and Darcy velocity variations

$$p = \frac{1}{\phi\beta_{\mathrm{f}}} P_0^{\mathrm{s}} \mathrm{e}^{i(\omega t - k_{\mathrm{s}} x)} + \frac{1}{\phi\beta_{\mathrm{f}}} P_0^{\mathrm{f}} \mathrm{e}^{i(\omega t - k_{\mathrm{f}} x)}, \tag{73}$$

$$u_2 = U_2^{\mathrm{s}} \mathrm{e}^{i(\omega t - k_{\mathrm{s}} x)} + U_2^{\mathrm{f}} \mathrm{e}^{i(\omega t - k_{\mathrm{f}} x)}. \tag{74}$$

Utilizing the first Equation (45), we obtain

$$\begin{aligned}
(1+R)U_1 &= U_2^{\mathrm{s}} + U_2^{\mathrm{f}},\\
\frac{ik_1}{\beta_1}(1-R)U_1 &= \frac{ik_2^{\mathrm{s}}}{\beta_2}U_2^{\mathrm{s}} + \frac{ik_2^{\mathrm{f}}}{\beta_2}U_2^{\mathrm{f}} + \frac{P_0^{\mathrm{f}} + P_0^{\mathrm{s}}}{\beta_{\mathrm{f}}}\\
0 &= i\omega(-\gamma_1 + \gamma_2\xi^{\mathrm{s}})U_2^{\mathrm{s}} + i\omega(-\gamma_1 + \gamma_2\xi^{\mathrm{f}})U_2^{\mathrm{f}}.
\end{aligned} \tag{75}$$

Further, by virtue of Equation (46), we get

$$\begin{aligned} -(1+R)U_1 &+ U_2^{\mathrm{s}} + U_2^{\mathrm{f}} = 0,\\ -\frac{k_1}{\beta_1}(1-R)U_1 &+ k_2^{\mathrm{s}}\left(\frac{1}{\beta_2}+\frac{\xi^{\mathrm{s}}}{\beta_{\mathrm{f}}}\right)U_2^{\mathrm{s}} + k_2^{\mathrm{f}}\left(\frac{1}{\beta_2}+\frac{\xi^{\mathrm{f}}}{\beta_{\mathrm{f}}}\right)U_2^{\mathrm{f}} = 0,\\ &(\gamma_1-\gamma_2\xi^{\mathrm{s}})U_2^{\mathrm{s}} + (\gamma_1-\gamma_2\xi^{\mathrm{f}})U_2^{\mathrm{f}} = 0. \end{aligned} \tag{76}$$

We assume zero attenuation in medium M_1, therefore $k_1 > 0$ is real and $\omega k_1 = v_1$ is the p-wave velocity in this medium. Note that v_1 is a characteristic of the medium M_1, which does not depend on the frequency.

Dividing through by U_1 and putting $Z_1 = R$, $Z_2 = U_2^{\mathrm{s}}/U_1$, and $Z_3 = U_2^{\mathrm{f}}/U_1$, we obtain the following system of equations

$$\begin{aligned} -Z_1 &+ Z_2 + Z_3 = 1,\\ \omega Z_1 &+ v_1 k_2^{\mathrm{s}}\left(\frac{\beta_1}{\beta_2}+\xi^{\mathrm{s}}\frac{\beta_1}{\beta_{\mathrm{f}}}\right)Z_2 + v_1 k_2^{\mathrm{f}}\left(\frac{\beta_1}{\beta_2}+\xi^{\mathrm{f}}\frac{\beta_1}{\beta_{\mathrm{f}}}\right)Z_3 = \omega,\\ &(\gamma_1-\gamma_2\xi^{\mathrm{s}})Z_2 + (\gamma_1-\gamma_2\xi^{\mathrm{f}})Z_3 = 0. \end{aligned} \tag{77}$$

Hence, using Equations (63) and (62) and notation (49), the system of equations (77) can be presented in the following asymptotic form

$$\begin{aligned} -Z_1 &+ Z_2 + Z_3 = 1,\\ \sqrt{\varepsilon}Z_1 &+ A_{22}Z_2 + A_{23}\sqrt{\varepsilon}Z_3 = \sqrt{\varepsilon},\\ &(A_{32}^{(1)} + A_{32}^{(2)} i\varepsilon)Z_2 + A_{33} i\varepsilon Z_3 = 0. \end{aligned} \tag{78}$$

The expressions for the coefficients A_{ij} can be obtained from the asymptotic formulae (53), (56), (57), (63), and (64):

$$A_{22} = \frac{v_1}{v_{\mathrm{b}}}\sqrt{\gamma_1+\gamma_2\gamma_v}\,\gamma_{\mathrm{s}}\frac{1-i}{\sqrt{2}}, \tag{79}$$

$$A_{23} = \frac{v_1}{v_{\mathrm{f}}}\sqrt{\frac{\gamma_2}{\gamma_1+\gamma_2\gamma_v}}\,\gamma_f, \tag{80}$$

$$A_{32}^{(1)} = \gamma_1+\gamma_2\gamma_v, \tag{81}$$

$$A_{32}^{(2)} = -\gamma_2\gamma_v\frac{1-\gamma_\varrho(\gamma_2\gamma_v+\gamma_1)}{\gamma_1+\gamma_2\gamma_v}, \tag{82}$$

$$A_{33} = -\frac{\gamma_\varrho\gamma_1-\gamma_1+\gamma_\varrho}{\gamma_1+\gamma_2\gamma_v}. \tag{83}$$

Here we used the notations

$$\gamma_{\mathrm{s}} = \beta_1\left(\frac{1}{\beta_2}-\gamma_v\frac{1}{\beta_{\mathrm{f}}}\right) \quad \text{and} \quad \gamma_{\mathrm{f}} = \beta_1\left(\frac{1}{\beta_2}+\frac{\gamma_1}{\gamma_2}\frac{1}{\beta_{\mathrm{f}}}\right). \tag{84}$$

From the last Equation (78)

$$Z_2 = -\frac{A_{33}}{A_{32}^{(1)}} i\varepsilon Z_3 + \cdots \tag{85}$$

This means that at low frequencies (i.e., at $\varepsilon \to 0$), the slow wave displacement is scaled with the velocity of fast displacement and, therefore, is one order of magnitude smaller. In other words, the slow part of the signal practically does not propagate and is mostly responsible for the reflection.

Substitution of (85) into the first two Equations (78) yields

$$\begin{aligned} -Z_1 \; + \; & \left(1 - \frac{A_{33}}{A_{32}^{(1)}} i\varepsilon\right) Z_3 = 1, \\ \sqrt{\varepsilon} Z_1 + & \left(A_{23}\sqrt{\varepsilon} - A_{22}\frac{A_{33}}{A_{32}^{(1)}} i\varepsilon\right) Z_3 = \sqrt{\varepsilon}. \end{aligned} \tag{86}$$

Cancelling the $\sqrt{\varepsilon}$ in the second Equation (86) and dropping terms of the order higher than $\sqrt{\varepsilon}$, we obtain that

$$Z_3 = Z_1 + 1. \tag{87}$$

Consequently

$$Z_1 = \frac{1 - A_{23} + A_{22}(A_{33}/A_{32}^{(1)}) i\sqrt{\varepsilon}}{1 + A_{23} - A_{22}(A_{33}/A_{32}^{(1)}) i\sqrt{\varepsilon}}. \tag{88}$$

Again, retaining only the terms of the order $\sqrt{\varepsilon}$, we finally obtain

$$Z_1 = \frac{1 - A_{23}}{1 + A_{23}} + \sqrt{2}\frac{\tilde{A}_{22} A_{33}}{A_{32}^{(1)}} \frac{1}{(1 + A_{23})^2}(1 + i)\sqrt{\varepsilon}, \tag{89}$$

where

$$\tilde{A}_{22} = \frac{v_1}{v_b}\sqrt{\gamma_1 + \gamma_2\gamma_v}\,\gamma_s. \tag{90}$$

Analysis of the expression (80) shows that in practical situations the coefficient A_{23} is greater than one. Therefore, the frequency-independent component of the reflection coefficient is negative. The frequency-dependent component of the reflection has the same sign as $\tilde{A}_{33}$. The latter is positive if and only if

$$\gamma_\varrho < \frac{\gamma_1}{1 + \gamma_1}. \tag{91}$$

The right-hand side of the last inequality is approximately equal to 0.5. Hence, roughly speaking, $\tilde{A}_{33}$ is positive when the fluid density is at least twice less than the bulk density of the saturated medium. In such a case the maximum of the absolute value of the reflection coefficient is attained at $\varepsilon = 0$. At the same time, for dense fluids, the first-order term of the asymptotic expansion, which is proportional to the square root of ε, may vanish and the first frequency-dependent term will be linear. In this case, the tortuosity coefficient becomes an important factor.

In the original variables (47), Equation (89) takes on the form

$$R = \frac{1 - A_{23}}{1 + A_{23}} + \sqrt{2}\frac{\tilde{A}_{22}A_{33}}{A_{32}^{(1)}}\frac{1}{(1 + A_{23})^2}(1 + i)\sqrt{\frac{\kappa \varrho_b}{\eta}\omega}. \tag{92}$$

Note that the last equation relates the reflectivity to the frequency through the factor of $\tau_D = \kappa \varrho_b / \eta$ having the dimension of time. It involves a property of the rock, the permeability coefficient, a property of the fluid, the viscosity, and a property of the coupled fluid–rock system, the bulk density. The frequency scaling proposed here is similar to but not the same as the scaling introduced in Geertsma and Smit (1961).

7. The Role of Relaxation Time and Tortuosity

The asymptotic calculations presented above show that the dimensionless parameter θ, related to both relaxation time and tortuosity factor, disappears from the first-order terms. However, if θ is large, then some expansions obtained in Sections 4 and 6 must be reviewed. Practically, the range of frequencies is limited by the specifications of the available tools. Therefore, it may happen that within the range of frequencies available for analysis the product $\theta\varepsilon$ is not negligibly small, and the passage to the limit as $\varepsilon \to 0$ should be replaced with analysis at some intermediate finite values of ε. In such a case, the asymptotic analysis must be performed differently. In this section, we consider two examples of such analysis.

First, let us assume that within the range of available frequencies, the parameter $\varepsilon\theta$ is of the order of one. In the original variables, this condition is equivalent to

$$\omega \sim \frac{1}{\tau}. \tag{93}$$

Regrouping the coefficients in Equation (50) and dividing through by $1 + i\theta\varepsilon$, we obtain

$$(A_0 + A_1^{\theta} i\varepsilon)\xi^2 + (B_0 + B_1^{\theta} i\varepsilon)\xi + C_0 + C_1^{\theta} i\varepsilon = 0, \tag{94}$$

where the coefficients with zero indices are the same as those in Equation (54), and

$$A_1^\theta = -\frac{\gamma_2\gamma_\varrho}{1+i\theta\varepsilon},$$
$$B_1^\theta = \frac{-1+\gamma_\varrho(1+\gamma_1)}{1+i\theta\varepsilon}, \tag{95}$$
$$C_1^\theta = \frac{\gamma_v\gamma_\varrho}{1+i\theta\varepsilon}.$$

Hence, the frequency-independent zero-terms of asymptotic expansions of the solutions ξ are the same as in Equation (51). To calculate the first-order coefficients, we note that formally the coefficients (95) are equal to the respective coefficients in Equations (54) evaluated at $\tau=0$ and divided by $1+i\theta\varepsilon$. This fact, in conjunction with the observation that the asymptotic expansion of the reflection coefficient (92) does not depend on τ, significantly simplifies the calculations. Indeed, for the first-order coefficients of asymptotic expansion for ξ we can reuse Equations (56) and (57) if we put there $\tau=0$ and multiply the right-hand sides by an additional factor of $1/1+i\theta\varepsilon$. Clearly, the calculations for the first-order terms of expansions of v and k can be carried out in a similar manner. The final result is that the reflection coefficient in the asymptotic expression (92) takes on the form

$$R=\frac{1-A_{23}}{1+A_{23}}+2\frac{A_{22}A_{33}}{A_{32}^{(1)}}\frac{1}{(1+A_{23})^2}\sqrt{i-\theta\varepsilon}\sqrt{\frac{\kappa\varrho_b}{\eta}\omega}. \tag{96}$$

Thus, if $\tau\omega = O(1)$, the relaxation time and tortuosity affect both the amplitude and the phase shift of the reflected signal.

Now, consider another extreme situation where $\theta \gg 1$, so that after the division of Equation (50) by θ all terms with θ in the denominator can be neglected. We obtain a quadratic equation

$$i\varepsilon(A_0\xi^2+B_0\xi+C_0)=0. \tag{97}$$

The latter implies that the frequency dependence of ξ (and, therefore, of the reflection coefficient as well) vanishes. Therefore, at a very large relaxation time (or, equivalently, at a very large tortuosity), the inertial term in Equation (39) makes the dissipation term on the right-hand side unimportant. Consequently, the fluid-saturated medium acts as an elastic composite medium and we arrive at a classical frequency-independent elastic wave reflection.

8. Conclusions

Equations of elastic wave propagation in fluid-saturated porous media can be obtained from the basic principles of filtration theory. Under different assumptions, these equations reduce either to Biot's poroelasticity model or to the pressure diffusion equation. Comparison between our derivation of poroelasticity equations and the original derivation by Biot shows that the tortuosity factor entering Biot's equations is proportional to the relaxation time from the dynamic version of Darcy's law. This result can be used to evaluate the tortuosity factor from a macroscopic flow experiment or microscopic-scale flow modeling (Patzek, 2001).

While, due to the high attenuation, slow poroelastic waves are rarely observed in practice, they significantly impact reflection–transmission processes making these processes frequency-dependent. This frequency dependence, in turn, affects both the amplitude and the phase of the reflected wave.

The low-frequency asymptotic behavior of the reflection of a plane seismic wave from an interface between an elastic medium and fluid-saturated porous medium has been investigated. In case of moderate tortuosity, the frequency-dependent component of the reflection coefficient is asymptotically proportional to the square root of the product of the reservoir fluid mobility and the frequency. If the tortuosity is extremely large, the possibility of which was demonstrated in Molotkov (1999), this scaling changes. In such a case, the frequency-dependent component of the reflection coefficient is more complicated and includes an additional factor depending on the dimensionless product of the relaxation time and the frequency of the signal.

The obtained results suggest that the nature of the frequency-dependence of the reflection coefficient is in viscous friction in fluid flow in the pore space, rather than in the contrast between the elastic properties of the overburden and reservoir rocks.

The obtained asymptotic reflection signal scaling has been successfully applied for imaging the productivity of hydrocarbon reservoir (Korneev *et al.* 2004).

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