Three-Phase Hydraulic Conductances in Angular Capillaries

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Abstract

In this paper, we extend to three fluid phases a prior finite-element study of hydraulic conductance of two-phase creeping flow in angular capillaries. Previously, we obtained analytic expressions for the hydraulic conductance of water in corner filaments. Here we present the results of a large numerical study with a high-resolution finite element method that solves the three-phase creeping flow approximation of the Navier-Stokes equation. Using the projection-pursuit regression approach, we provide simple analytic expressions for the hydraulic conductance of an intermediate layer of oil sandwiched between water in the corners of the capillary and gas in the center. Our correlations are derived for the oil layers bounded by the concave or convex interfaces that are rigid or allow perfect slip. Therefore, our correlations are applicable to drainage, spontaneous imbibition, and forced imbibition with maximum feasible hysteresis of each contact angle, oil/water and gas/oil. These correlations should be useful in pore-network calculations of three-phase relative permeabilities of spreading oils. Finally, we compare our results with the existing correlations by Zhou et al., and Hui & Blunt, who assumed thin-film flow with an effective film thickness proportional to the ratio of the flow area to the length of the no-flow boundary. On average, our correlations are two-four times closer to the numerical results than the corresponding correlations by Zhou et al., and Hui & Blunt.

Introduction

Since direct measurement of flow of three immiscible fluids is very difficult, the pore-scale models of three-fluid systems2,5,12,13,17 have blossomed. One of the more important advancements in such models was the approximation of single pore geometries as angular capillaries with square or arbitrary triangular cross-sections. Although real pores are not exactly square or triangular, this approximation allows one to capture the flow of water in the pore corners and the flow of oil and gas in the pore center. As illustrated in Figure 1a, when three fluids are moving in a single angular capillary, the most wetting fluid (water or Fluid 1) resides in the corner and the most nonwetting fluid (gas or Fluid 3) fills the center. The third fluid (oil or Fluid 2) forms an intermediate layer sandwiched between the other two fluids. In some cases of large contact angles and positive spreading coefficients, we may find more than one sandwiched layer, Figure 1b. These intermediate layers are a few micrometers thick and have been observed in micromodel experiments.5,11,16,21 It is drainage through these oil layers that is responsible for the high oil recoveries seen experimentally.5,11,16 Although it was initially thought that only spreading oils could form such layers in angular pores, it has been theoretically predicted and experimentally verified that nonspreading oil can also form intermediate layers in the crevices of the pore space.3,11,25 Therefore, the formation of sandwiched layers is not only related to the positive spreading coefficient, but also depends on the curvatures of the o/w and g/o interfaces, the corner geometry and the contact angles.3

Creeping flow of oil in these intermediate layers is the subject of this paper. In particular, we study the hydraulic conductances of oil flow in stable fluid layers of different sizes and geometries. We provide simple and accurate correlations for these conductances by relating them to the interface geometry, fluid contact angles, and pore geometry. The proposed correlations should be useful in pore network calculations of three-phase relative permeabilities.

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A similar approach has been successful in single-phase\textsuperscript{19} and two-phase\textsuperscript{18} flow in angular capillaries. Assuming that equilibrium o/w and g/o interfaces are stable in creeping flow\textsuperscript{1,20} one can solve the Navier-Stokes equation in the intermediate oil layer, given its fixed boundaries. From the solution of the Navier-Stokes equation, the average flow velocity of fluid $i$, ($v_i$), is calculated, and its hydraulic conductance, $g_i$, is estimated from a linear relation between the volumetric flow rate in the layer, $Q_i$, and the gradient of the total driving force per unit area, $\Xi_i$:

\[ Q_i = (v_i) A_i = g_i \Xi_i \]  

(1)

where $A_i$ is the layer cross-sectional area.

Zhou et al.\textsuperscript{24} presented an approximate analytical solution for oil flow along a layer sandwiched between water and air in angular capillaries and derived expressions for flow resistance and hence hydraulic conductance with no slip and perfect slip conditions at the interfaces. Zhou et al. derived their expressions by assuming thin-film flow with an effective film thickness proportional to the ratio of the flow area to the length of the no-flow boundary. Their expressions were derived for the zero oil/water contact angle, and are limited to the oil flow in the layers bounded by concave menisci, as in drainage, but not in forced imbibition. For non-zero oil/water contact angles and convex interfaces, Hui & Blunt\textsuperscript{8,9} proposed a modified version of Zhou et al. expressions. Later in this paper, we will discuss in some detail the relative accuracy of all these expressions.

The objective of this paper is twofold: (1) develop a numerical algorithm that solves the velocity distribution and hence hydraulic conductance in three-phase flow with various geometries and interface boundary conditions, and (2) provide simple and accurate correlations for the hydraulic conductances of the intermediate layers of oil sandwiched between water and gas. The paper is organized as follows. First, we calculate the various geometrical descriptors of the oil layer such as its area, perimeter, and shape factor. Second, we present the mathematical formulation of the boundary value problems in creeping flow. Third, we describe the finite element approximations of these problems, and discuss the numerical results. Finally, we correlate the layer hydraulic conductance with the layer geometry, the pore corner geometry, and the oil/water and gas/oil contact angles.

### Layer Geometry and Stability

Before simulating fluid flow in the sandwiched layer, we study its geometry and stability. As shown in Figure 2, the formation of the layer depends on five parameters: corner half-angle $\beta$, o/w contact angle $\theta_{21}$, g/o contact angle $\theta_{32}$, o/w meniscus-apex distance $b_1$, and g/o meniscus-apex distance $b_2$. Zhou et al.\textsuperscript{24} and Hui & Blunt\textsuperscript{8} define their hydraulic conductance expressions using the radius of meniscus curvature instead of the meniscus-apex distance. The radius of curvature and the meniscus-apex distance are related by Equation 2, where $r_1$ and $r_2$ are the radii of curvature of the o/w and g/o interfaces, respectively.

\[ b_1 = r_1 \frac{\cos(\theta_{21} + \beta)}{\sin(\beta)} \quad b_2 = r_2 \frac{\cos(\theta_{32} + \beta)}{\sin(\beta)} \]  

(2)

The g/o interface is always concave (i.e., $\theta_{32} + \beta < \pi/2$). For $\theta_{32} + \beta \geq \pi/2$, the layer becomes unstable and cannot form. On the other hand, the o/w interface can be concave ($\theta_{21} + \beta < \pi/2$), flat ($\theta_{21} + \beta = \pi/2$), or convex ($\theta_{21} + \beta > \pi/2$), Figure 2. We shall now define the dimensionless meniscus-apex distances $b_1$ and $b_2$ by scaling the spatial coordinates with the g/o meniscus-apex distance, $b_2$. As a result, the dimensionless distance $b_1$ is 1 and the dimensionless distance $b_2$ is $b_1/b_2$.

The three relevant descriptors of the layer geometry are its dimensionless area, circumference, and shape factor. First, however, we define the three constants that will be repeatedly used in the calculations:

\[ E_0^{ij} = \frac{\pi}{2} - \theta_{ij} - \beta \]
\[ E_1^{ij} = \frac{\cos(\theta_{ij} + \beta)}{\sin \beta} \]
\[ E_2^{ij} = \frac{\cos(\theta_{ij} + \beta)}{\sin \beta} \cos \theta_{ij} \]  

(3)

The layer dimensionless cross-sectional area, $\hat{A}_L$, is defined as follows:

\[ \hat{A}_L = \begin{cases} \frac{(E_2^{32} - E_2^{30})}{(E_1^{32})^2} - \left( \frac{b_1}{b_2} \right)^2 \sin \beta \cos \beta & \text{if } \theta_{21} + \beta = \pi/2 \\ \frac{(E_2^{32} - E_0^{32})}{(E_1^{32})^2} - \left( \frac{b_1}{b_2} \right)^2 \left( \frac{E_2^{32} - E_0^{32}}{(E_1^{32})^2} \right) & \text{otherwise} \end{cases} \]  

(4)

The layer dimensionless circumference, $\hat{P}_L$, is defined as

\[ \hat{P}_L = \left( 2 \left( 1 - \frac{b_1}{b_2} \right) + \hat{L}_{21} + \hat{L}_{32} \right) \]

(5)

where $\hat{L}_{21}$ is the dimensionless length of the o/w interface calculated with

\[ \hat{L}_{21} = \begin{cases} \frac{2b_1}{b_2} \sin \beta & \text{if } \theta_{21} + \beta = \pi/2 \\ \frac{b_1 E_0^{31}}{b_2 E_1^{31}} & \text{otherwise} \end{cases} \]  

(6)

and similarly the dimensionless length of the g/o interface, $\hat{L}_{32}$, is calculated by
Figure 1: Different configurations of water, 1, oil, 2, and gas, 3, in a single triangular pore

\[ \beta_{\text{min}} + \beta < \pi / 2 \]

\[ \beta_{\text{min}} + \beta = \pi / 2 \]

\[ \beta_{\text{min}} + \beta > \pi / 2 \]

Figure 2: Sandwiched layers with concave, flat, or convex o/w interface

\[ L_{32} = 2 \frac{E_{32}^{g/o}}{E_{1}^{g/o}} \]

Then, the layer normalized shape factor, \( \hat{G}_L \), is defined as the ratio of the layer dimensionless area and square of the layer dimensionless circumference:

\[ \hat{G}_L = \frac{A_L}{(P_L)^2} \]

The layer actual area is calculated as \( A_L = (b_2)^2 \hat{A}_L \), and the actual perimeter is given by \( P_L = b_2 \hat{P}_L \). The actual and normalized shape factors are equal. Finally, the layer cannot exist, because the two menisci touch, if,

\[
\begin{aligned}
&\frac{b_1}{b_2} \geq \cos^2 \beta \left[ 1 - \frac{\cos \beta}{E_{32}^{g/o}} + \frac{\cos \beta \sin \theta_{32}}{E_{1}^{g/o}} \right] \text{ if } \theta_{21} + \beta = \pi / 2 \\
&\frac{b_1}{b_2} \geq \min \left\{ 1, \frac{|E_{32}^{g/o} \cos \theta_{32} - \sin \beta|}{|E_{1}^{g/o} \cos \theta_{21} - \sin \beta|} \right\} \text{ otherwise}
\end{aligned}
\]

\[ (9) \]

Mathematical Formulation

Assuming steady state, creeping isothermal flow of Newtonian, incompressible and constant viscosity fluids, the combined continuity and Navier-Stokes equations that describe the flow reduce to an elliptic Poisson equation:

\[ \nabla^2 v_i = -\frac{\Xi_i}{\mu_i} \quad \forall (x_1, x_2) \in \Omega_i \]

where \( i = 1, 2, 3 \) denotes water, oil, and gas, respectively, \( v_i \) is the \( i \)th fluid velocity, \( \mu_i \) is the \( i \)th fluid viscosity, \( x_1 \) and \( x_2 \) are the spatial coordinates across the capillary, Figure 3, and \( \Xi_i \) is the gradient of the total driving force per unit area defined as

\[ \Xi_i = -\nabla p_i + \rho_i f \]

where, \( \nabla p_i \) is the pressure gradient in fluid \( i \), \( \rho_i \) is the \( i \)th fluid density, and \( f \) is the body force per unit mass. We also assume that the interfacial tensions are constant, and the buoyancy forces are negligible in the capillary, i.e., both Bond numbers are much less than one.

Using a scaling scheme similar to that implemented in Ref.,\textsuperscript{18} we scale the spatial coordinates with the g/o meniscus-apex distance, \( b_2 \), and the fluid velocities with the oil viscosity, \( \mu_2 \), and the respective gradient of the force, \( \Xi_2 \), driving oil flow:

\[ \hat{x}_j = \frac{x_j}{b_2} \quad j = 1, 2, 3 \]

\[ \hat{v}_i = \frac{v_i}{b_2} \mu_2 \]
Spatial coordinates of the sandwiched layer: the flow is along the duct in the $x_3$-direction and the velocity distribution is calculated in the $x_1$ and $x_2$ directions; $\Gamma_s$ denotes the duct wall, whereas, $\Gamma_{21}$, and $\Gamma_{32}$ denote the o/w and g/o interfaces, respectively.

After scaling, the dimensionless form of Equation 10 is:

$$\tilde{\mu}_i \nabla^2 \tilde{v}_i = -1$$

$$\tilde{\mu}_1 = \frac{\mu_1}{\mu_2}, \quad \tilde{\mu}_2 = 1, \quad \tilde{\mu}_3 = \frac{\mu_3}{\mu_2}$$  \hspace{1cm} (13)

Although Equations 10 and 11 are applicable to any of the three fluids, we focus here on the oil in the sandwiched layer. Our formulation is incomplete without specifying the boundary conditions for this layer. As shown in Figure 3, we need to impose boundary conditions along the duct walls, $\Gamma_s$, the o/w interface, $\Gamma_{21}$, and the g/o interface, $\Gamma_{32}$.

We impose a no-slip boundary condition along the duct walls:

$$\tilde{v}_i = 0 \text{ on } \Gamma_s$$  \hspace{1cm} (14)

We consider two different boundary conditions along the o/w and g/o interfaces. The first boundary condition requires infinite surface shear viscosity of the interface, which becomes a surfactant-laden rigid wall with a no-slip condition:

$$\tilde{v}_i = 0 \text{ on } \Gamma_{21} \text{ or } \Gamma_{32}$$  \hspace{1cm} (15)

The second boundary condition describes a perfect-slip condition with zero surface shear viscosity of the interface. This condition translates mathematically to:

$$\nabla \tilde{v}_i \cdot \mathbf{n}_i = 0 \text{ on } \Gamma_{21} \text{ or } \Gamma_{32}$$  \hspace{1cm} (16)

where, $\mathbf{n}_i$ is the unit outward normal vector along the interface. These two interface boundary conditions result in four possible configurations of boundary conditions:

BC-1: no-slip o/w interface and no-slip g/o interface

BC-2: no-slip o/w interface and perfect-slip g/o interface

BC-3: perfect-slip o/w interface and no-slip g/o interface

BC-4: perfect-slip o/w interface and perfect-slip g/o interface

Another possible interface boundary condition assumes continuity of velocity and shear stress along the interface.

$$\nabla \tilde{v}_2 \cdot \mathbf{n}_2 = \tilde{\mu}_1 \nabla \tilde{v}_1 \cdot \mathbf{n}_1 \text{ on } \Gamma_{21}$$

$$\nabla \tilde{v}_2 \cdot \mathbf{n}_2 = \tilde{\mu}_3 \nabla \tilde{v}_3 \cdot \mathbf{n}_3 \text{ on } \Gamma_{32}$$  \hspace{1cm} (17)

This last boundary condition is not considered here. Finally, there is a no-flow boundary condition along the lines of symmetry (see the discussion below).

**Numerical Approximation**

We solve the boundary value problem (13)-(16) numerically using the Finite Element Method (FEM). The FEM solution was obtained using the MATLAB partial differential equation (PDE) toolbox. This Toolbox provides a powerful and flexible environment for the study and solution of partial differential equations in two space dimensions and time. We now present examples of mesh generation, solution visualization, and convergence.

**Mesh Generation**

Accuracy of the FE solution depends on how well the computational domain is discretized. A good numerical solution requires a mesh that follows the shape of the duct walls and the curvatures of the interfaces. The PDE toolbox has many powerful built-in functions that enable the user to generate the finite element meshes with required properties. In our simulations, we have used the built-in MATLAB function *initmesh* to initiate the first discretization of the domain. Then, the initial mesh is refined five more times using the function *refinemesh*. In each refinement, each triangle in the mesh is divided into four new triangles. Because of the layer symmetry with respect to the corner angle bisector, the FEM analysis is performed on 1/2 of the domain. Depending on the layer geometry, the maximum number of elements generated for a layer (half-domain) in our simulations was 109,824 elements, whereas the minimum number of elements was 5,376 elements. The average number of elements of all the cases considered in this analysis was 20,446 elements. In Figure 4, we present examples of layer meshes for two geometrical configurations. The number of elements shown in the plot is just for illustration, the meshes used in the numerical analysis were much finer.

**Dimensionless Velocity Profile**

Once we have generated a mesh and specified appropriate boundary conditions, we solve Equation 13 in order to calculate the profile of the dimensionless velocity in the layer. In Figure 5, we show the dimensionless velocity profiles in a specific layer for the four boundary conditions: BC-1, BC-2, BC-3, and BC-4. The layer dimensionless area, $A_L$, the circumference, $P_L$, and the shape factor $G_L$, are 0.1901, 3.7913, and 0.0132, respectively. Benefiting from the symmetry of the layer
with respect to the corner angle bisector, the FEM analysis is performed using 16,896 elements on 1/2 of the layer; the second half mirrors the first one. From Figure 5, one can see that the interface boundary conditions influence significantly the velocity profile and hence the average velocity and hydraulic conductance of the layer. The average dimensionless velocity in the layer, \( \langle \tilde{v}_L \rangle \), is calculated as follows:

\[
\langle \tilde{v}_L \rangle = \frac{\sum_{k=1}^{N} \tilde{v}_k \tilde{A}_k}{\sum_{k=1}^{N} \tilde{A}_k} \quad (18)
\]

where, \( N \) is the total number of elements in the layer, \( \tilde{A}_k \) is the element \( k \) dimensionless area, and \( \tilde{v}_k \) is the element \( k \) dimensionless velocity calculated at the center of the element. The average dimensionless velocities in the layers shown in Figure 5 are 0.0016, 0.0053, 0.0027, and 0.0163, respectively, with the boundary conditions BC-1, BC-2, BC-3, and BC-4.

The dimensionless hydraulic conductance, \( \tilde{g}_L \), of the layer is

\[
\tilde{g}_L = 2 \sum_{k=1}^{N} \tilde{v}_k \tilde{A}_k 
\]

(19)

The factor of 2 compensates for the calculations on 1/2 of the layer. For the layers in Figure 5, the dimensionless hydraulic conductances are 0.000302, 0.0010, 0.000513, and 0.0031, respectively, for BC-1, BC-2, BC-3, and BC-4.

Noticing that \( A_L = (b_2)^2 \tilde{A}_L \), and using Equations 1 and 12, we can relate the layer actual hydraulic conductance, \( g_L \), to the dimensionless conductance, \( \tilde{g}_L \), as follows:

\[
g_L = \frac{(b_2)^4}{\mu_2} \tilde{g}_L \quad (20)
\]

Convergence To test the convergence of our FEM solution, we calculate the ratio of the hydraulic conductances, \( \tilde{g}_L^{N_1} \) and \( \tilde{g}_L^{N_2} \), calculated with the number of finite elements \( N_1 \) and \( N_2 = 4N_1 \). This ratio approaches one when the solution is converging. In Figure 6, we show an example of this calculation for BC-2: no-slip o/w interface and perfect-slip g/o interface. Similar results were obtained for the other three boundary conditions. From Figure 5, one can see that with few thousands of elements, about 6,000, the numerical solution converges. Similar observation holds for simulations with the other boundary conditions.

Results

Deriving universal curves that approximate hydraulic conductances of a sandwiched layer requires intensive simulations of different layer geometries, set by different combinations of the parameters \( \beta, \theta_{21}, \theta_{32} \), and \( b_1/b_2 \). In this analysis we have generated 17,167 stable layer geometries with arbitrary values of \( \beta = 5^\circ - 85^\circ, \theta_{21} = 0^\circ - 170^\circ, \theta_{32} = 0^\circ - 80^\circ, \) and \( b_1/b_2 = 0.1 - 0.9 \). For each generated layer, the dimensionless hydraulic conductance, \( \tilde{g}_L \), is calculated for the four boundary conditions shown in Figure 5.

In this section, we investigate what are the effects of the layer dimensionless parameters \( \beta, \theta_{21}, \theta_{32} \), and \( b_1/b_2 \) on its dimensionless hydraulic conductance, \( \tilde{g}_L \). All four boundary conditions presented in Figure 5 exhibited similar behavior; therefore, we restrict our description to one boundary configuration, BC-2: no-slip o/w interface and perfect-slip g/o interface. Same conclusion can be applied to the other three boundary conditions.

Figure 7 relates the logarithm of the dimensionless hydraulic conductance to the corner half-angle \( \beta \), o/w contact angle \( \theta_{21} \), and g/o contact angle \( \theta_{32} \) for a fixed ratio \( b_1/b_2 \). For small corner half-angles, \( \beta < 10^\circ \), the variation of the hydraulic conductance with \( \theta_{21} \) and \( \theta_{32} \) is relatively minor. However, as the corner half angle increases, \( \beta \geq 10^\circ \), the hydraulic conductance varies considerably with \( \theta_{21} \) and \( \theta_{32} \). As shown in Figure 8, for a fixed \( \beta \), increasing \( \theta_{21} \) decreases the layer hydraulic conductance,
Figure 5: Velocity distribution in a sandwiched layer ($\beta = 60^\circ$, $\theta_{21} = 90^\circ$, $\theta_{32} = 10^\circ$, and $b_1/b_2 = 0.25$) with different interface boundary conditions: BC-1, BC-2, BC-3 and BC-4, respectively.

Figure 6: Convergence of the FEM solution for eight different layers with BC-2: no-slip o/w interface and perfect-slip g/o interface.
whereas increasing $\theta_{32}$ increases it. Also, both contact angles, $\theta_{21}$ and $\theta_{32}$, affect the hydraulic conductance considerably. The effect of the ratio $b_1/b_2$ on the layer hydraulic conductance is shown in Figure 9. One may notice that increasing $b_1/b_2$ results in a large variation of the hydraulic conductance. Smaller variation is observed when $b_1/b_2$ is less than 0.4.

Figures 7-9 demonstrate that different combinations of $\beta$, $\theta_{21}$, $\theta_{32}$, and $b_1/b_2$ may result in a huge variation of the layer hydraulic conductance. They also foreshadow the difficulty in creating a straightforward, easy-to-use relationship among these parameters and the layer hydraulic conductance. As our analysis shows, there is no straightforward relation between the layer conductance and the layer geometrical features such as its area, perimeter, or shape factor. Therefore, a rigorous statistical procedure is required to derive accurate correlations for the layer hydraulic conductance.

**Universal Curves for the Layer Hydraulic Conductance**

Here we attempt to derive the universal curves that approximate the hydraulic conductance of the sandwiched layer. As shown above, regular statistical procedures may not describe adequately the nonlinear interactions among the dimensionless dependent variable $Y = (\tilde{g}_L)$ and any of the layer dimensionless predictor variables

\[ X = (\beta, \theta_{21}, \theta_{32}, b_1/b_2, \tilde{P}_L, \tilde{A}_L, L_{21}, L_{32}, \tilde{G}_L). \]

Therefore, we choose projection-pursuit regression to obtain the universal curves for the layer hydraulic conductances. Projection-pursuit regression is a computer-intensive procedure that applies an additive model to the projected variables:

\[ \theta(Y) = \alpha_0 + \sum_{j=1}^{M} \lambda_j f_j(\alpha_k X) + \epsilon \]  

(21)

where $\theta(Y)$ is any transformation of the dependent variable $Y$, $\alpha_0$ is the mean of $\theta(Y)$, $\lambda_j$ and $\alpha_k$ are the constants calculated by the regression model, $f_j$ are the functions fitted by the model, $M$ is a dimension chosen by the user, and $\epsilon$ is the deviation of the fitted values from the corresponding true ones. Thus projection-pursuit regression uses an additive model on the predictor variables which are formed by “projecting” matrix $X$ in $M$ carefully chosen directions. The “pursuit” part indicates that an optimization technique is used to find “good” direction vectors, $\alpha_k$. The statistical language S-PLUS is used to implement the projection-pursuit regression.

Using projection-pursuit regression, the dimensionless hydraulic conductance in the layer is given by

\[ \ln(\tilde{g}_L) = \alpha_0 + \lambda_1 f_1(z) \]  

(22)

where $\alpha_0$, $\lambda_1$, $z$, and $f_1(z)$ depend on the interfaces boundary conditions as follows:

**BC-1: No-Slip o/w Interface and No-Slip g/o Interface**

\[
\begin{align*}
\alpha_0 &= -7.9998 \\
\lambda_1 &= 1.7474 \\
\lambda_2 &= 0.0797 \ln(\beta) + 0.5540 \ln(\tilde{G}_L) + \\
&+ 0.7698 \ln(\tilde{A}_L) - 0.1494 \theta_{32} - 0.2679 \frac{b_1}{b_2} \\
f_1(z) &= 1.3062z + 4.9465
\end{align*}
\]  

(23)

**BC-2: No-Slip o/w Interface and Perfect-Slip g/o Interface**

\[
\begin{align*}
\alpha_0 &= -7.2153 \\
\lambda_1 &= 1.8026 \\
\lambda_2 &= 0.3227 \ln(\beta) + 0.5948 \ln(\tilde{G}_L) + \\
&+ 0.7141 \ln(\tilde{A}_L) - 0.1516 \theta_{32} - 0.0959 \frac{b_1}{b_2} \\
f_1(z) &= 0.0008541z^2 + 1.2363z + 4.9495
\end{align*}
\]  

(24)

**BC-3: Perfect-Slip o/w Interface and No-Slip g/o Interface**

\[
\begin{align*}
\alpha_0 &= -7.5325 \\
\lambda_1 &= 1.6435 \\
\lambda_2 &= 0.1367 \ln(\beta) + 0.4339 \ln(\tilde{G}_L) + \\
&+ 0.7444 \ln(\tilde{A}_L) - 0.1740 \theta_{32} + 0.4568 \frac{b_1}{b_2} \\
f_1(z) &= 1.5380z + 4.7172
\end{align*}
\]  

(25)

**BC-4: Perfect-Slip o/w Interface and Perfect-Slip g/o Interfaces**

\[
\begin{align*}
\alpha_0 &= -6.4543 \\
\lambda_1 &= 1.7558 \\
\lambda_2 &= 0.4005 \ln(\beta) + 0.14981 \ln(\tilde{G}_L) + \\
&+ 0.3587 \ln(\tilde{A}_L) + 0.7446L_{21} - 0.3169L_{32} \\
f_1(z) &= 0.0409z^5 + 0.4377z^4 + \\
&+ 1.7096z^3 + 3.0028z^2 + 4.1682z + 3.8056
\end{align*}
\]  

(26)

In Figures 10, 12, 14 and 16, we plot the deviations of the hydraulic conductances calculated with the projection-pursuit regression from the corresponding FEM solutions, for the four boundary conditions, BC-1, BC-2, BC-3 and BC-4, respectively. The relative error is calculated as follows:

\[
\text{Relative Error} = \frac{\tilde{g}_L - \tilde{g}_L^{FE}}{\tilde{g}_L^{FE}} \times 100
\]  

(27)

where $\tilde{g}_L^{FE}$ is the hydraulic conductance calculated by the finite-element model. The mean absolute relative errors of these approximations are 9.5%, 9.8%, 13.9%, and 18.0% for BC-1, BC-2, BC-3, and BC-4, respectively. The actual conductance is calculated with Equation 20.
Figure 7: The logarithm of the dimensionless hydraulic conductance vs. corner half-angles $\beta$ and contact angles $\theta_{21}$ and $\theta_{32}$ for $b_1/b_2 = 0.8$, and with BC-2.

Figure 8: The logarithm of the dimensionless hydraulic conductance vs. contact angles $\theta_{21}$ and $\theta_{32}$ for $b_1/b_2 = 0.8$ and $\beta = 15^\circ$, and with BC-2.
The most widely used expressions for the hydraulic conductance of the sandwiched layers were derived by Zhou et al.\textsuperscript{24} Zhou’s layer dimensionless hydraulic conductance for \( \theta_{32} + \beta < \pi/2 \) and \( \theta_{21} = 0^\circ \) is given by

\[
\tilde{g}_L = \frac{\Phi}{\Psi} = \tilde{A}_c \phi_3^2 (1 - \sin \beta)^2 \times \\
\times \left( \phi_2 \cos \theta_{32} - \phi_1 - \cot \beta (1 - \phi_3) \tilde{R}_o \right)^3
\]

\[
\Psi = 12 \sin^2 \beta (1 - \phi_3)^2 (\phi_2 \cos \theta_{32} - \phi_1)^2 \times \\
\times \left( \phi_2 + f_1 \phi_1 - \cot \beta (1 - f_2 \phi_3) \tilde{R}_o \right)^2 \tag{28}
\]

where the constants \( \phi_1, \phi_2, \phi_3, \tilde{A}_c, \tilde{R}_o \) are as follows:

\[
\begin{align*}
\phi_1 &= \frac{\pi}{2} - \theta_{32} \\
\phi_2 &= \cot \beta \cos \theta_{32} - \sin \theta_{32} \\
\phi_3 &= \left( \frac{\pi}{2} - \beta \right) \tan \beta 
\end{align*}
\]

\[
\tilde{A}_c = \left( \frac{\sin \beta}{\cos (\theta_{32} + \beta)} \right)^2 \left[ \cos \theta_{32} (\cot \beta \cos \theta_{32} - \sin \theta_{32}) + \\
+ \theta_{32} + \beta - \frac{\pi}{2} \right] \tag{30}
\]

\[
\tilde{R}_o = \frac{b_1 \cos (\theta_{32} + \beta)}{b_2 \cos (\theta_{21} + \beta)} \tag{31}
\]

The quantity \( f \) is used to indicate the boundary condition at the Fluid/Fluid interface. A value of \( f = 1 \) represents a no-slip boundary, while a value of \( f = 0 \) is a perfect-slip boundary. \( f_1 \) is the boundary condition at g/o interface and \( f_2 \) is the boundary condition at the o/w interface. For non-zero o/w contact angles, or for convex interfaces, Hui & Blunt\textsuperscript{8,9} proposed the following modification to Equation 28:

\[
\tilde{g}_L = \frac{\tilde{A}_o \phi_3^2 (1 - \sin \beta)^2 \tan \beta}{12 \tilde{A}_c \sin^2 \beta (1 - \phi_3) \left[ 1 + f_1 \phi_3 - (1 - f_2 \phi_3) \sqrt{\frac{\tilde{A}_o}{\tilde{A}_c}} \right]^2} \tag{32}
\]

where \( \tilde{A}_w \) and \( \tilde{A}_o \) are calculated by

\[
\begin{align*}
\tilde{A}_w &= \left( \frac{b_1 \sin \beta}{b_2 \cos (\theta_{21} + \beta)} \right)^2 \left[ \cos \theta_{21} (\cot \beta \cos \theta_{21} - \sin \theta_{21}) + \\
+ \theta_{21} + \beta - \frac{\pi}{2} \right] \\
\tilde{A}_o &= \tilde{A}_c - \tilde{A}_w \tag{33}
\end{align*}
\]

Deviations of the hydraulic conductances calculated with Equations 28 and 32 from the corresponding FEM solutions with the four boundary conditions in this analysis are shown in Figures 11, 13, 15 and 17. The points with light colors were generated with the Zhou et al. approximation (28) for the zero o/w contact angle, whereas the dark-color points belong to the Hui & Blunt approximation (32) for non-zero o/w contact angles. The mean absolute relative errors of these approximations are 22.2%, 24.1%,

Figure 9: The logarithm of the dimensionless hydraulic conductance vs. the ratio \( b_1/b_2 \) and contact angles \( \theta_{21} \) and \( \theta_{32} \) for \( \beta = 15^\circ \), and with BC-2
28.0%, and 64.9% for BC-1, BC-2, BC-3, and BC-4, respectively. Thus, the projection-pursuit regression method gives results better than the expressions proposed by Zhou et al., and Hui & Blunt.

**Conclusions**

A numerical approach has been developed to obtain the hydraulic conductance of an intermediate fluid layer sandwiched between two other fluids. This approach has led to the simple and accurate correlations of the layer hydraulic conductance with the various layer geometry descriptors, and with four different boundary conditions:

1. Expressions for calculating the layer perimeter, area, shape factor, and stability have been derived from the corner half-angle $\beta$, the interface contact angles $\theta_{21}$ and $\theta_{32}$, and the meniscus-apex distances $b_1$ and $b_2$.

2. A standard finite element method has been used to solve numerically Poisson’s equation for creeping, isothermal flow in an intermediate fluid layer formed in a corner of a polygonal capillary. The finite elements have been implemented using the PDE toolbox in MATLAB®.

3. Each flow domain was discretized with a mesh that followed the shape of the capillary walls and the curvatures of the interfaces. In this analysis, we have generated 17,167 different layers with average number of elements equal to 20,452. The minimum number of elements in a layer was 5,376, while the maximum number of elements was 109,312. The convergence analysis indicated that for few thousands of elements, about 6,000, the FEM solution converged to machine precision.


5. The dependence of the dimensionless hydraulic conductance on the layer dimensionless parameters $\beta$, $\theta_{21}$, $\theta_{32}$, and $b_1/b_2$ has been discussed. In general, the dimensionless hydraulic conductance decreases with the decreasing $\beta$, increasing $\theta_{21}$, decreasing $\theta_{32}$, and increasing $b_1/b_2$. Different combinations of these parameters result in a huge variability of the layer flow conductance.

6. Projection-pursuit regression has been used to obtain the universal correlations of the layer hydraulic conductance with the relevant descriptors of the layer geometry. These expressions have been listed for each boundary condition described in Item 4.

7. The hydraulic conductance expressions proposed here have been compared with the expressions proposed by Zhou et al., and Hui & Blunt. We have compared the relative deviation of the conductance calculated with each method from the corresponding finite element solution. We have shown that the mean absolute relative errors of the projection-pursuit regression expressions are 9.5%, 9.8%, 13.9%, and 18.0% for the four boundary conditions, respectively. The corresponding mean absolute errors of the Zhou et al., and Hui & Blunt expressions are 22.2%, 24.1%, 28.0%, and 64.9%. Thus, the expressions proposed here are better than those proposed by Zhou et al., and Hui & Blunt, by a factor 2-4.

8. Overall, our correlations of the layer hydraulic conductances are simple and can be used with confidence in the computationally intensive two-phase and three-phase pore-network simulations of drainage and imbibition with contact line pinning and contact angle hysteresis.

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**References**


### Nomenclature

- **A** = cross-sectional area, $L^2$
- **\( \tilde{A} \)** = dimensionless area
- **\( b_1 \)** = oil/water meniscus-apex distance, $L$
- **\( b_2 \)** = gas/oil meniscus-apex distance, $L$
- **\( E_i \)** = geometrical constant
- **\( f \)** = body force per unit mass, $FM^{-1}$
- **\( f_j \)** = $j^{th}$ statistical approximation function
- **\( g \)** = hydraulic conductance, $L^6F^{-1}T^{-1}$
- **\( \tilde{g} \)** = dimensionless hydraulic conductance
- **\( G \)** = shape factor
- **\( \tilde{G} \)** = normalized shape factor
- **\( \tilde{L}_{21} \)** = dimensionless length of oil/water interface
- **\( L_{32} \)** = dimensionless length of gas/oil interface
- **\( n \)** = unit outward normal vector
- **\( P \)** = circumference, $L$
- **\( \tilde{P} \)** = dimensionless circumference
- **\( r_1 \)** = radius of curvature of oil/water interface, $L$
- **\( r_2 \)** = radius of curvature of oil/water interface, $L$
- **\( Q \)** = volumetric flow rate $L^3T^{-1}$
- **< v >** = average flow velocity, $LT^{-1}$
- **\( v \)** = velocity, $LT^{-1}$
- **\( X \)** = vector of independent variables
- **\( Y \)** = dependent variable
- **\( z \)** = statistical fit variable
- **\( \alpha \)** = statistical fit constant
- **\( \beta \)** = corner half-angle
- **\( \Gamma_i \)** = $i^{th}$ boundary of flow domain
- **\( \epsilon \)** = approximation error
- **\( \theta_{21} \)** = oil/water contact angle
- **\( \theta_{32} \)** = gas/oil contact angle
- **\( \phi \)** = transformation of dependent variable
- **\( \mu \)** = viscosity, $FL^{-2}T$
- **\( \lambda \)** = statistical fit constant
- **\( \rho \)** = density, $ML^{-3}$
- **\( \nabla p \)** = pressure gradient, $FL^{-3}$
- **\( \Sigma \)** = gradient of driving force per unit area, $FL^{-3}$
Figure 10: Relative deviation of projection-pursuit regression flow conductance from the corresponding FEM solution for BC-1 (the mean absolute error is 9.5%).

Figure 11: Relative deviation of Zhou et al. and Hui & Blunt flow conductance from the corresponding FEM solution for BC-1: (the mean absolute error is 22.2%).
Figure 12: Relative deviation of projection-pursuit regression flow conductance from the corresponding FEM solution for BC-2 (the mean absolute error is 9.8%).

Figure 13: Relative deviation of Zhou et al. and Hui & Blunt flow conductance from the corresponding FEM solution for BC-2 (the mean absolute error is 24.1%).
Figure 14: Relative deviation of projection-pursuit regression flow conductance from the corresponding FEM solution for BC-3 (the mean absolute error is 13.9%).

Figure 15: Relative deviation of Zhou et al. and Hui & Blunt flow conductance from the corresponding FEM solution for BC-3 g/o interface (the mean absolute error is 28.0%).
Figure 16: Relative deviation of projection-pursuit regression flow conductance from the corresponding FEM solution for BC-4 (the mean absolute error is 18.0%).

Figure 17: Relative deviation of Zhou et al. and Hui & Blunt flow conductance from the corresponding FEM solution for BC-4 (the mean absolute error is 64.9%).