Transport in Porous Media **57:** 125–152, 2004.

Relative Permeabilities for Strictly Hyperbolic Models of Three-Phase Flow in Porous Media

RUBEN JUANES[1] and TADEUSZ W. PATZEK[2,3]
[1]*Department of Petroleum Engineering, Stanford University, Stanford, CA 94305, U.S.A.*
[2]*Department of Civil and Environmental Engineering, University of California, Berkeley, CA 94720, U.S.A.*
[3]*Lawrence Berkeley National Laboratory, Earth Sciences Division, Berkeley, CA 94720, U.S.A.*

(Received: 16 September 2002; in final form: 29 January 2004)

Abstract. Traditional mathematical models of multiphase flow in porous media use a straightforward extension of Darcy's equation. The key element of these models is the appropriate formulation of the relative permeability functions. It is well known that for one-dimensional flow of three immiscible incompressible fluids, when capillarity is neglected, most relative permeability models used today give rise to regions in the saturation space with elliptic behavior (the so-called elliptic regions). We believe that this behavior is not physical, but rather the result of an incomplete mathematical model. In this paper we identify necessary conditions that must be satisfied by the relative permeability functions, so that the system of equations describing three-phase flow is strictly hyperbolic everywhere in the saturation triangle. These conditions seem to be in good agreement with pore-scale physics and experimental data.

Key words: three-phase flow, relative permeability, hyperbolic system, conservation laws, elliptic regions, Oak experiments.

1. Introduction

Mathematical modeling of multiphase flow in porous media is, to a large extent, still an open issue. In our opinion, one of the main difficulties stems from the fact that different processes dominate at different scales: capillary forces dominate at the pore scale, while viscous and/or gravity forces dominate at the field scale. As a result, the development of continuum theories of multiphase flow has proven to be an exceptionally challenging task.

The key ingredients of traditional formulations of multiphase flow are mass conservation equations, and a multiphase form of Darcy's equation. Darcy's equation is an approximate form of the fluid momentum balance in creeping flow through porous media. This *postulate* is supported for single-phase flow by experimental evidence and by volume averaging as a first-order approximation (Hassanizadeh, 1986). On the other hand, the usual multiphase flow extension of Darcy's equation due to Muskat (1949) does *not* emanate from averaging of the microscopic equations of multiphase systems (Hassanizadeh and Gray, 1993). A number of inconsistencies of the standard formulation have been identified, and alternative approaches have been proposed (Barenblatt *et al.*, 1990; Avraam and Payatakes, 1995; Gray and Hassanizadeh, 1998), which are yet to be fully explored.

However, there is still a great interest in Darcy-like formulations, as they are almost universally used in hydrogeology and petroleum engineering. In this framework, success of the formulation depends heavily on the use of 'correct' relative permeabilities. Traditionally, they are taken as functions of current fluid saturations alone. This *very* strong assumption does not account for: (1) *hysteretic effects*, which include the past saturation history into the formulation (Lenhard and Parker, 1987); (2) *nonequilibrium effects*, which introduce the concept of a relaxation time for pore-scale rearrangement of fluid saturations (Barenblatt *et al.*, 1990; Silin and Patzek, 2004); and (3) *the flow regime* determined by the ratios of viscous, capillary, and gravity forces, which influences the pore-scale mechanisms of fluid displacement (Lenormand *et al.*, 1988; Avraam and Payatakes, 1995; Yortsos *et al.*, 1997)

Here we study one-dimensional horizontal flow through porous media of three immiscible, incompressible fluids, by using the common multiphase extension of Darcy's equation. This model leads to a 2×2 system of saturation equations in which capillarity enters as a nonlinear diffusion term. The analysis of the character of this system in the limit of negligible capillarity was first addressed in the Russian literature. Charny (1963) and Stklyanin (1960) pointed out the possibility that the system may be of mixed elliptic/hyperbolic type, but concluded that it is hyperbolic for physically realistic flows. In the Western literature, it was first shown by Bell *et al.* (1986) that a common relative permeability model could lead to regions in the saturation space where the system is elliptic. It is now known that virtually all relative permeability models used today give rise to elliptic regions (Fayers, 1987; Shearer, 1988; Shearer and Trangenstein, 1989; Holden, 1990; Hicks Jr. and Grader, 1996). What is more, the mathematical analysis of Shearer (1988), Shearer and Trangenstein (1989), and Holden (1990), seems to suggest that elliptic regions are an unavoidable feature of three-phase flow models. Under the conditions assumed in these analyses, the only relative permeability models which do not display an elliptic region are those where the relative permeability of each phase is a function of its own saturation only, usually referred to as 'Corey-type' models (Marchesin and Medeiros, 1989; Trangenstein, 1989). In this case, the elliptic region shrinks to an isolated *umbilic point*, where the system is *nonstrictly* hyperbolic. However, measured relative permeabilities – especially of the intermediate wetting phase – never exhibit dependence on their respective saturations only (Leverett and Lewis, 1941; Oak, 1990).

We find mixed elliptic/hyperbolic behavior disturbing for many reasons, and are of the opinion that elliptic regions are the artifacts of an incomplete mathematical model. The objective of this paper is to show that it is possible to impose conditions on the relative permeabilities that preserve strict hyperbolicity of the three-phase flow equations, even if the usual extension of Darcy's equation is employed. In doing so, we acknowledge that a Darcy-type formulation is unable to reproduce the complex physics of three-phase flow. Consequently, the relative permeabilities should be understood as *functionals*, which depend not only on the fluid saturations, but also on the wettability properties and the viscosity ratios. General analytical solutions to the three-phase flow equations can then be obtained (Juanes and Patzek, 2004).

In Section 2 we present the mathematical model of three-phase flow, and introduce the classification of the governing system of equations. In Section 3 we derive necessary conditions that the relative permeability functions must satisfy for the system of equations to be strictly hyperbolic for all admissible saturation states. We show, in Section 4, that the essential condition that needs to be imposed agrees well with experimental data. In Section 5 we draw the main conclusions, and anticipate ongoing and future research.

2. Traditional Displacement Theory of Three-Phase Flow

2.1. SYSTEM OF GOVERNING EQUATIONS

We study three-phase flow (of water, oil and gas) in porous media under the following assumptions: (1) one-dimensional flow; (2) immiscible fluids; (3) incompressible fluids; (4) homogeneous rigid porous medium; (5) multiphase flow extension of Darcy's law; (6) negligible mass forces including gravitational effects; (7) negligible capillary pressure effects; and (8) negligible non-equilibrium effects. When the fractional flow formalism is used, the governing equations are a *pressure* equation, and a 2×2 system of *saturation* equations. Using appropriately rescaled space and time variables, the system of saturation equations is expressed as:

$$\partial_t \boldsymbol{u} + \partial_x \boldsymbol{f} = \boldsymbol{0}, \tag{1}$$

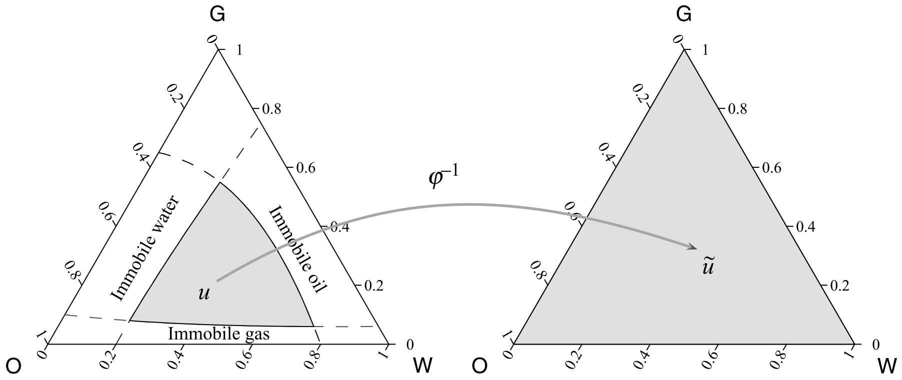
where

$$\boldsymbol{u} := \begin{pmatrix} u \\ v \end{pmatrix} \equiv \begin{pmatrix} S_w \\ S_g \end{pmatrix}, \qquad \boldsymbol{f} := \begin{pmatrix} f \\ g \end{pmatrix} \equiv \begin{pmatrix} f_w \\ f_g \end{pmatrix}, \tag{2}$$

are the vectors of water and gas saturations, and water and gas fractional flows, respectively. When gravity and capillary forces are not considered, the fractional flow function of the α-phase is simply $f_\alpha = \lambda_\alpha/\lambda_T$, where λ_α is the relative mobility of the α-phase, and $\lambda_T := \lambda_w + \lambda_o + \lambda_g$ is the total mobility. The relative mobility is defined as $\lambda_\alpha := k_{r\alpha}/\mu_\alpha$, where $k_{r\alpha}$ is the relative permeability and μ_α is the dynamic viscosity of phase α.

2.2. FLOW REGIONS AND REDUCED SATURATIONS

Experimental evidence suggests that there is a threshold saturation for each phase, below which that phase is immobile (Wyckoff and Botset, 1936; Leverett and



(a) Space of *actual* saturations (b) Space of *reduced* saturations

Figure 1. Schematic of the map between (a) the space of *actual* saturations, and (b) the space of *reduced* saturations.

Lewis, 1941). As a result, three-phase flow takes place only in a region inside the saturation triangle. The nature of these threshold saturations depends on the wettability of the fluids, and on the displacement process (Geffens *et al*., 1951). Here, we use the term 'immobile' saturation, and assume that appropriate values are used for each fluid and for the particular process (saturation path) of interest. In principle, these threshold or endpoint saturations do not have to be constant (Fayers and Matthews, 1984; Fayers, 1987).

The three-phase flow region in the space of *actual* saturations $\boldsymbol{u}$ can be mapped onto the entire ternary diagram of *reduced* saturations $\tilde{\boldsymbol{u}}$, as shown in Figure 1. We assume that the map $\boldsymbol{u} = \boldsymbol{\varphi}(\tilde{\boldsymbol{u}})$ is C^1 invertible and orientation-preserving, so that by simple change of variables we can study three-phase flow in terms of reduced saturations:

$$\partial_t \tilde{\boldsymbol{u}} + \partial_x \tilde{\boldsymbol{f}} = \mathbf{0}, \tag{3}$$

where the newly defined flux $\tilde{\boldsymbol{f}}$ is related to the original flux function as follows:

$$\tilde{\boldsymbol{f}}(\tilde{\boldsymbol{u}}) := (\boldsymbol{\varphi}')^{-1} \boldsymbol{f}(\boldsymbol{\varphi}(\tilde{\boldsymbol{u}})). \tag{4}$$

The relative permeabilities and, consequently, fractional flows, are expressed in Equation (3) as functions of the reduced saturations. We assume that an appropriate functional form is used, that is consistent with the rock, fluid, and process descriptors. To simplify notation we shall drop the tildes from Equation (3) and write

$$\partial_t \boldsymbol{u} + \partial_x \boldsymbol{f} = \mathbf{0}, \tag{5}$$

but still refer to the system in terms of the reduced saturations.

2.3. CHARACTER OF THE SYSTEM OF EQUATIONS

We write the system (5) in quasilinear form:

$$\partial_t \boldsymbol{u} + \boldsymbol{A}(\boldsymbol{u})\partial_x \boldsymbol{u} = \boldsymbol{0}, \tag{6}$$

where

$$\boldsymbol{A}(\boldsymbol{u}) := \boldsymbol{f}'(\boldsymbol{u}) \equiv \mathrm{D}_{\boldsymbol{u}}\boldsymbol{f} \equiv \begin{pmatrix} f_{,u}(\boldsymbol{u}) & f_{,v}(\boldsymbol{u}) \\ g_{,u}(\boldsymbol{u}) & g_{,v}(\boldsymbol{u}) \end{pmatrix} \tag{7}$$

is the Jacobian matrix of the system at the saturation state $\boldsymbol{u}$. Subscripts after a comma denote differentiation (e.g. $f_{,u} \equiv \partial_u f$). The classification of the system (6) reduces to analyzing the behavior of the eigenvalue problem

$$\boldsymbol{Ar} = \nu \boldsymbol{r}, \tag{8}$$

where the Jacobian matrix $\boldsymbol{A}$, the eigenvalue ν, and the right eigenvector $\boldsymbol{r}$, are evaluated at a state $\boldsymbol{u}$. For the eigenvalue problem (8) with a 2×2 real matrix, it is well-known (Coddington and Levinson, 1955) that there exists a real nonsingular matrix $\boldsymbol{T}$ such that, after the change of variables $z = \boldsymbol{Tr}$, the equivalent eigenvalue problem

$$(\boldsymbol{TAT}^{-1})z = \nu z \tag{9}$$

has a *real* coefficient matrix $\boldsymbol{J} := \boldsymbol{TAT}^{-1}$, which has one of the following canonical forms:

1. $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \lambda \neq \mu,$
2. $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \quad \beta \neq 0,$
3. $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$
4. $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$

These four canonical forms provide the basis for the classification of the system of first-order partial differential equations. Following the terminology in Zauderer (1983), we denote the system whose Jacobian matrix has the canonical form of cases 1 through 4 above, respectively:

1. *Strictly hyperbolic.* The eigenvalue problem has two real, distinct eigenvalues. The Jacobian matrix is diagonalizable and there are two real and linearly independent eigenvectors (Zauderer, 1983).
2. *Elliptic.* The eigenvalues are complex conjugates, and there are no real characteristic curves that may act as carriers of possible discontinuities in the solution (Zauderer, 1983).

3. *Nonstrictly hyperbolic.* There is a double real eigenvalue, and the Jacobian matrix is diagonalizable. Every direction is characteristic, and the system is hyperbolic (real eigenvalues and linearly independent eigenvectors) but not strictly hyperbolic (which requires that the eigenvalues be distinct).
4. *Parabolic.* The system has a real, double eigenvalue, and the Jacobian matrix is defective (non-diagonalizable). There is only one eigenvector and, therefore, there is only one real characteristic direction.

For 2×2 systems with real coefficients, this classification is general. The *eigenvalues* ν_i ($i = 1, 2$), of the original Jacobian matrix (7) are given by

$$\nu_{1,2} = \frac{1}{2}\left[f_{,u} + g_{,v} \mp \sqrt{(f_{,u} - g_{,v})^2 + 4 f_{,v} g_{,u}} \right]. \tag{10}$$

The physical interpretation of the eigenvalues (when they are real) is the characteristic speeds at which waves describing changes in saturation propagate through the domain. The *right eigenvectors* $\boldsymbol{r}_i = [r_{iu}, r_{iv}]^{\mathrm{t}}$ ($i = 1, 2$), are calculated by the following expressions:

$$\frac{r_{1v}}{r_{1u}} = \frac{\nu_1 - f_{,u}}{f_{,v}} = \frac{g_{,u}}{\nu_1 - g_{,v}}, \tag{11}$$

$$\frac{r_{2u}}{r_{2v}} = \frac{f_{,v}}{\nu_2 - f_{,u}} = \frac{\nu_2 - g_{,v}}{g_{,u}}. \tag{12}$$

When they are real, the right eigenvectors correspond to the directions (in the phase space) of admissible changes in saturation.

3. Relative Permeabilities for Strict Hyperbolicity

3.1. LOSS OF STRICT HYPERBOLICITY IN TRADITIONAL MODELS

Charny (1963) pointed out the possibility that the system (5) may not be hyperbolic for all saturation states. He concluded, with reference to Stklyanin (1960), that in physically realistic three-phase flows, the system is hyperbolic. Independently, Bell *et al.* (1986) observed that Stone I relative permeabilities (Stone, 1970) gave rise to elliptic regions inside the saturation triangle. In subsequent publications (Fayers, 1987; Shearer, 1988; Shearer and Trangenstein, 1989; Holden, 1990; Hicks Jr. and Grader, 1996), it was shown that occurrence of elliptic regions is the rule rather than exception for the most common relative permeability models.

Loss of strict hyperbolicity of three-phase flow models was analyzed by Shearer (1988), Shearer and Trangenstein (1989), and Holden (1990). They also used reduced saturations, and therefore limited their analysis to the three-phase flow region. This limitation is not particularly restrictive, for it can be shown that elliptic regions cannot exist in the one-phase and two-phase flow regions (Falls and Schulte, 1992). Their analysis is based on the behavior of the system of equations on the three sides of the saturation triangle. It must be emphasized that such analysis is meaningful, because it provides information about the direction of eigenvectors associated with the slow and fast waves when one phase is nearly absent. Below we give a brief summary of this analysis (see Juanes, 2003 for a more detailed discussion).

Shearer (1988) and Holden (1990) start by assuming the behavior of relative permeabilities in *two-phase flow*. Relative mobilities of both phases (say, water and gas) are *assumed* to have a zero value and a *zero derivative* at their endpoint saturations. This behavior, which is taken for granted without further discussion, is then used as a guidance to impose conditions on the edges of the saturation triangle for the *three-phase flow* case.

Shearer (1988) imposes the following conditions on the edges and corners of the saturation triangle:

1. *Consistency conditions* ('(B-L) conditions'), which reduce three-phase relative mobilities to the assumed behavior for two-phase flow, when one of the phases is not mobile.
2. A *first interaction condition* ('(I1) condition'), which limits the effect of the immobile phase on the flow, compared with that of the other two phases. This condition, together with the consistency conditions above, forces the right eigenvector associated with the fast characteristic speed, $\boldsymbol{r}_2$, to be parallel to the edge.
3. A *second interaction condition* ('(I2) condition'), makes the right eigenvector associated with the fast characteristic, $\boldsymbol{r}_2$, point *into* the saturation triangle for states $\boldsymbol{u}$ near a vertex. A sufficient (but not necessary) condition that satisfies this requirement is that the off-diagonal terms of the Jacobian matrix $\boldsymbol{A}(\boldsymbol{u})$, are positive.

Holden (1990) imposes very similar conditions. The following is required on the edges and near the corners of the saturation triangle:

1. The *value* of the relative mobility of a phase is zero along the edge of zero reduced saturation of that phase.
2. The *derivative* of the relative mobility of a phase along the *normal direction* to the edge of zero reduced saturation is also zero. We note that the conditions above on the normal derivatives imply that the '(I1) interaction condition' of Shearer (1988) is immediately satisfied and, as a result, the eigenvector of the fast family is parallel to the edges of the saturation triangle.
3. Three possible types of behavior near the vertices were considered, based on the sign of the off-diagonal terms of the Jacobian matrix:

   (A1) Both are positive: $f_{,v} > 0,\ g_{,u} > 0$.
   (A2) Have different signs: $f_{,v} g_{,u} < 0$.
   (A3) Both are negative: $f_{,v} < 0,\ g_{,u} < 0$.

   Using a wettability argument, it is suggested that condition (A1) is the most reasonable at all three corners. This condition implies that the '(I2) interaction condition' of Shearer (1988) is automatically satisfied.

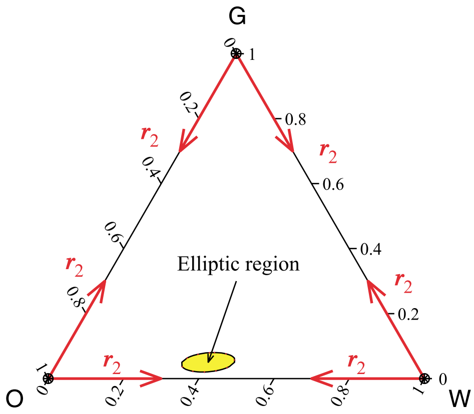


Figure 2. Direction of fast eigenvectors $\boldsymbol{r}_2$ along the edges of the saturation triangle for the models analyzed by Shearer and Holden.

We summarize the conditions imposed by Shearer (1988) and Holden (1990) as follows (see Figure 2):

1. The right eigenvector associated with the fast characteristic family, $\boldsymbol{r}_2$, is *parallel* to the edges of the triangle of reduced saturations.
2. The fast eigenvector $\boldsymbol{r}_2$ points *into* the triangle for saturation states near the vertices.

The *assumed* behavior at the edges and corners of the saturation triangle has a profound impact on the character of the system. The first consequence is that each vertex of the saturation triangle is an umbilic point. The second consequence is that, in general, an elliptic region must exist inside the saturation triangle. This general result can be proved using ideas of projective geometry (Shearer, 1988).

Naturally, the question of whether elliptic regions in the saturation space are physically plausible arises. We briefly point out some of the reasons why local elliptic behavior seems to be an artifact of the mathematical model, rather than a necessary consequence dictated by physics.

1. Equation (5) is a system of first-order equations in *space-time* coordinates. Thus, the physical meaning of a system with mixed elliptic/hyperbolic behavior is very different from that when the independent variables are two *space* coordinates, such as in *steady* transonic flow (Keyfitz, 1990). In the former case, 'initial data' should be imposed in such a way that the principle of causality[1] is not violated (Fayers, 1987).
2. Saturation states inside the elliptic region give rise to linearly ill-posed problems. More precisely, a bounded solution to the linearized Cauchy problem

[1] "The causality of natural processes may be interpreted as implying that the conditions in a body at time t are determined by the past history of the body, and that no aspect of its future behavior need to be known in order to determine all of them" (Truesdell and Noll, 1965).

(initial value problem on an unbounded domain) does not exist when arbitrarily close – but not equal – asymptotic left and right states are inside the elliptic region. This fact is in frontal disagreement with the notion of three-phase flow displacement, where one expects a bounded transition between the right (initial) state and the left (injected) state.

3. Appropriate entropy conditions have not yet been found so as to allow both existence *and* uniqueness of solutions (Azevedo and Marchesin, 1995). Although it is unclear how nonlinear effects impact the ill-posedness of the problem (Holden *et al.*, 1990; Azevedo and Marchesin, 1995), we agree completely with Azevedo *et al.* (2002) in that "both linear instability and nonuniqueness are outside the range of 'good' behavior for hyperbolic conservation laws."
4. More specifically to models of three-phase flow in porous media, it has been found (Hicks Jr. and Grader, 1996) that different models matching experimental data equally well (or equally badly), produce elliptic regions in opposite corners of the saturation triangle. This result suggests a nonphysical arbitrariness to the location of elliptic behavior in phase space for traditional models of three-phase flow.
5. If capillarity is introduced in the formulation and a traveling wave solution is sought for the Riemann problem (unbounded domain, and piecewise constant initial data with a single discontinuity), the critical points of the associated 2×2 dynamical system are *spiral points* (Coddington and Levinson, 1955). If a traveling wave solution exists, it will necessarily present a spiral-like behavior near the critical points, which translates into oscillatory (nonmonotonic) saturation profiles. Validity of this type of solution is questionable on several counts: (1) As the capillarity effects are taken to zero, oscillations collapse into a singular shock, of dubious physical interpretation; (2) Introducing 'sufficient' capillarity does not cure the problem of oscillatory behavior, as the spiral-like orbit persists asymptotically. In fact, it has been shown that the effects of capillarity may actually enlarge the locus of instability, and extend it to the strictly hyperbolic region (Azevedo *et al.*, 2002).
6. Numerical simulations seem to corroborate, at least in first instance, the nonphysical behavior of solutions inside the elliptic region (Bell *et al.*, 1986; Jackson and Blunt, 2002). For arbitrarily close left and right states inside the elliptic region, the solution develops wildly oscillatory waves, which are not observed in experiments. Moreover, the wave pattern is unstable with respect to the initial states and the gridblock size as the mesh is refined (Bell *et al.*, 1986). We emphasize that the type of instability alluded to here is completely different from the physical instability of immiscible displacements.

From the observations above, it is difficult to justify the physical relevance of elliptic regions (Fayers, 1987; Shearer and Trangenstein, 1989; Trangenstein, 1989; Sahni *et al.*, 1996; Azevedo *et al.*, 2002). An attempt in this direction is the recent paper by Jackson and Blunt (2002), who used a serial model of bundle of capillary tubes to argue that elliptic regions exist in a simplistic but physically realizable porous medium. They assumed that relative permeabilities are fixed functions of saturations, and investigated the character of the system for large values of the gravity number. However, the constraints imposed by the authors on the communication between the bundles reduce their model to a *single* bundle of capillary tubes, which cannot reproduce the physics of a dynamic immiscible displacement (e.g. the formation of an oil bank). We are of the opinion that elliptic regions are mere artifacts of an incomplete mathematical model. Inappropriateness of the formulation may have several sources but, in the context of Darcy-type formulations, the element of the formulation that first needs to be revisited is the relative permeability model.

The only relative permeability models which do not produce elliptic regions (under the assumed behavior at the edges and corners) are models of 'Corey-type' (Marchesin and Medeiros, 1989; Trangenstein, 1989). For models of this type, the elliptic region shrinks to an isolated umbilic point, and the system is nonstrictly hyperbolic. Solutions to such systems have been studied extensively (see Guzmán and Fayers, 1997; Marchesin and Plohr, 2001 and the references therein), and have been used in the interpretation of displacement experiments (Grader and O'Meara Jr., 1988; Sahni *et al.*, 1996). However, the assumption that the relative permeability of each phase depends *only* on the saturation of that phase is overly restrictive, and it is not supported by direct measurements and pore-scale considerations. It may be possible to reduce the elliptic region to an isolated umbilic point for more general relative permeability models (Holden, 1990). However, this requires a continuous deformation of the relative permeability functions, which is physically unappealing.

In order to allow for more general relative permeability models that do not produce elliptic regions, we revisit the assumed behavior at the edges of the saturation triangle. In fact, it is widely recognized that the slope of experimental relative permeabilities near the endpoints is often ill-defined (Fayers, 1987).

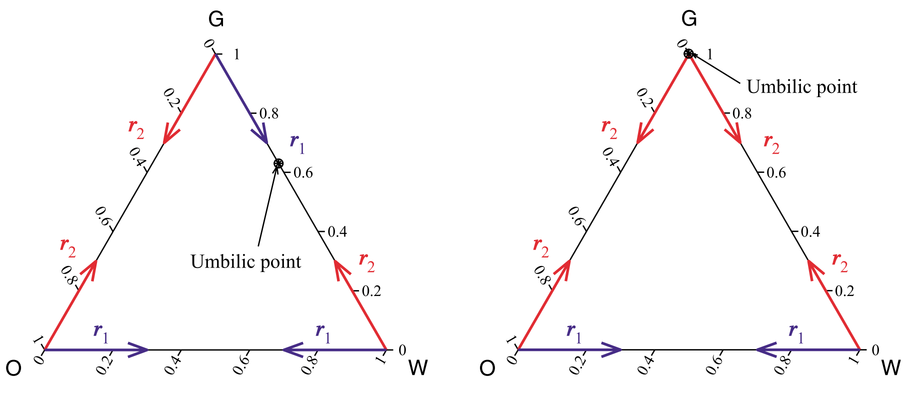
3.2. CONDITIONS FOR STRICT HYPERBOLICITY

The generic approach in the existing literature can be summarized as follows: a certain behavior of the relative permeabilities is *assumed*, and loss of strict hyperbolicity inside the saturation triangle is *inferred*. We adopt the opposite viewpoint: we assume that the system is strictly hyperbolic, and investigate the conditions on the relative permeabilities such that strict hyperbolicity is preserved. In doing so, we keep the condition that the mobility of a phase is identically equal to zero along the edge of zero reduced saturation of that phase. The direct consequence of this condition is that the edges of the saturation triangle are invariant lines for the system (5), that is, if a phase is not present initially, it will remain absent. Such consistency condition is required for the three-phase system to describe the two-phase Buckley–Leverett equation when the third phase is not present. Mathematically, the zero mobility condition implies that on each edge of the saturation triangle one eigenvector is *always* parallel to that edge.

It is easy to see that the requirement above (one eigenvector parallel to each edge), precludes the possibility of having a strictly hyperbolic system everywhere along the edges of the ternary diagram. There are two different ways in which strict hyperbolicity may fail on the boundary of the saturation triangle:

1. *The system is strictly hyperbolic at all three vertices.* For vertices to be strictly hyperbolic, eigenvectors lying on each of the two edges must be of different family (e.g. at the O corner, $\boldsymbol{r}_1$ is parallel to the OW edge and $\boldsymbol{r}_2$ is parallel to the OG edge). But then, there must exist at least one edge that has a parallel eigenvector of the fast family near one vertex, and of the slow family near the other vertex. Inevitably, an umbilic point – where characteristic speeds of the slow and fast characteristic families coincide – must exist somewhere on this edge, because eigenvectors do not rotate along the edge (Figure 3(a)).
2. *At least one of the vertices is an umbilic point.* As we show below, it is possible to have a model that is nonstrictly hyperbolic at the G vertex and strictly hyperbolic everywhere else (Figure 3(b)).

Having the considerations above in mind, the key observation is that, whenever gas is present as a continuous phase, the mobility of gas is usually much higher than that of the other two fluids (water and oil). To honor this physical behavior, we associate fast characteristic paths with displacements involving changes in gas saturation, even in the region of small gas saturation. The immediate consequence is that the eigenvector associated with the fast family of characteristics ($\boldsymbol{r}_2$) is *transversal* – and not parallel – to the oil–water edge of the ternary diagram (Figure 3(b)). As we shall see, this conceptual picture permits that the system will be strictly hyperbolic everywhere *inside* the saturation triangle. The G vertex, correspond-



(a) Umbilic point on WG edge (b) Umbilic point at G vertex

Figure 3. Direction of fast ($\boldsymbol{r}_2$) and slow ($\boldsymbol{r}_1$) eigenvectors along the edges of the saturation triangle for: (a) models with strictly hyperbolic vertices; (b) models of the type we propose.

ing to 100% reduced gas saturation, remains an umbilic point because fast paths corresponding to the OG and WG edges coalesce.

Let us recapitulate the conceptual picture expressed in Figure 3(b):

1. Along the oil–water (OW) edge, the eigenvector associated with the *slow* characteristic family ($\boldsymbol{r}_1$) is parallel to the edge. The system is strictly hyperbolic everywhere along the edge, including the O and W vertices.
2. Along the oil–gas (OG) and water–gas (WG) edges, the eigenvector associated with the *fast* characteristic family ($\boldsymbol{r}_2$) is parallel to these edges. The system is strictly hyperbolic everywhere along the edges except at the G vertex, which is an umbilic point.

Below we present a systematic study of the necessary conditions required for strict hyperbolicity of the system. On each edge we identify two types of conditions: *Condition I* enforces that eigenvectors of the appropriate family are parallel to the edge, and *Condition II* guarantees strict hyperbolicity of the system along the edge.

3.2.1. *Analysis Along the OW Edge*

The slow eigenvector being parallel to the OW edge ($\boldsymbol{r}_1 = [1, 0]^{\mathrm{t}}$) implies:

$$g_{,u} = 0. \tag{13}$$

When expressed in terms of mobilities, Condition I above reads:

$$\lambda_{g,u} = 0. \tag{14}$$

This condition is immediately satisfied for any model, as the gas mobility is identically zero along this edge. It is also necessary that the denominator in Equation (11) be different from zero:

$$\nu_1 - g_{,v} \neq 0. \tag{15}$$

Introducing (13) in (10), together with condition (15), yields:

$$\nu_1 = f_{,u}, \qquad \nu_2 = g_{,v}. \tag{16}$$

The condition for strict hyperbolicity, $\nu_1 < \nu_2$, implies that

$$H_{ow} := g_{,v} - f_{,u} > 0, \tag{17}$$

which is equivalent to

$$\lambda_{g,v} > \lambda_{w,u} - \lambda_{T,u}\frac{\lambda_w}{\lambda_T}. \tag{18}$$

Condition II above is the fundamental requirement for strict hyperbolicity of the system of equations of three-phase flow. When this condition is evaluated at the vertices of the OW edge, one obtains:

$$\lambda_{g,v} \;>\; \lambda_{w,u} \quad \text{at the O vertex}, \tag{19}$$

$$\lambda_{g,v} \;>\; -\lambda_{o,u} \quad \text{at the W vertex}, \tag{20}$$

Table I. Summary of conditions along the OW edge

Condition	Fractional flows		Mobilities
I	$g_{,u} = 0$	$\Leftrightarrow$	$\lambda_{g,u} = 0$
II	$g_{,v} - f_{,u} > 0$	$\Leftrightarrow$	$\lambda_{g,v} > \lambda_{w,u} - \lambda_{T,u}\frac{\lambda_w}{\lambda_T}$
II at O			$\lambda_{g,v} > \lambda_{w,u}$
II at W			$\lambda_{g,v} > -\lambda_{o,u}$

where the inequalities above are strict. In particular, Equations (19)–(20) dictate that the gas relative permeability must *not* have a zero derivative at its endpoint saturation. A summary of the conditions at the OW edge is given in Table I. We make the following important *remarks*:

1. The requirement of a nonzero endpoint-slope of the gas relative permeability is a *necessary* condition for strict hyperbolicity, which is violated by the models of all previous studies on this subject.
2. This behavior of gas relative permeability is in good agreement with experimental observations of two-phase (Wyckoff and Botset, 1936; Geffens *et al.*, 1951; Osoba *et al.*, 1951) and three-phase flow (Leverett and Lewis, 1941; Oak, 1990; Oak *et al.*, 1990), both in drainage and imbibition. We demonstrate this agreement in Section 4.
3. A finite positive slope for the gas relative permeability can also be justified from the point of view of pore-scale processes, using a wettability argument. As an illustration, we consider a bundle of cylindrical capillary tubes, and assume that water is the most wetting, and gas the least wetting phase. For expositional simplicity, we discuss the case of a uniform distribution of radii between $r_{\min}$ and $r_{\max}$, although the argument holds for virtually any other pore size distribution. To account for the corners and crevices of real porous media – where water is present in the form of filaments – we take $r_{\min} = 0$. At any given state of fluid saturations, gas will occupy the capillaries of larger radii, and water the smaller capillaries. It is a simple exercise to show that the relative permeabilities of water, oil and gas are given by:
$$k_{rw} = S_w^{5/3}, \qquad k_{ro} = (1 - S_g)^{5/3} - S_w^{5/3}, \qquad k_{rg} = 1 - (1 - S_g)^{5/3}. \tag{21}$$
The water relative permeability has a zero endpoint-slope ($k'_{rw} = 0$ at $S_w = 0$), while the gas relative permeability has a positive endpoint-slope ($k'_{rg} = 5/3$ at $S_g = 0$). We certainly do not defend the capillary bundle model as a realistic representation of the porous medium, but just bring attention to the fact that even in this simplistic model, the conditions derived above are satisfied.

3.2.2. *Analysis Along the OG Edge*

A necessary condition for the fast eigenvector to be parallel to the OG edge ($\boldsymbol{r}_2 = [0, 1]^{\mathrm{t}}$) is:

$$f_{,v} = 0. \tag{22}$$

In terms of mobilities, Condition I reads:

$$\lambda_{w,v} = 0. \tag{23}$$

This condition is immediately satisfied because the water mobility is identically zero along this edge. The denominator of Equation (12) must be different from zero:

$$\nu_2 - f_{,u} \neq 0. \tag{24}$$

Using (22) and (24), the eigenvalues take the following expressions along the OG edge:

$$\nu_1 = f_{,u}, \qquad \nu_2 = g_{,v}. \tag{25}$$

We impose that the system be strictly hyperbolic everywhere along the OG edge, excluding the G vertex. The condition of strict hyperbolicity, $\nu_1 < \nu_2$, implies that (Condition II):

$$H_{og} := g_{,v} - f_{,u} > 0, \tag{26}$$

or, equivalently:

$$\lambda_{g,v} > \lambda_{w,u} + \lambda_{T,v}\frac{\lambda_g}{\lambda_T}. \tag{27}$$

When we specialize Condition II at the O vertex, we recover condition (19), which requires that the gas relative permeability must have a positive slope at its endpoint saturation. The G vertex is assumed to be an umbilic point, where the slow and fast characteristic speeds coincide, that is, $\nu_1 = \nu_2$. When expressed in terms of relative mobilities, the condition reads:

$$\lambda_{w,u} + \lambda_{o,v} = 0. \tag{28}$$

The conditions at the OG edge are summarized in Table II.

3.2.3. *Analysis Along the WG Edge*

The analysis at the WG edge is complicated by the fact that it is a tilted line in the (u, v)-plane. The fast eigenvector will be parallel to the WG edge ($\boldsymbol{r}_2 = [-1, 1]^{\mathrm{t}}$) if:

$$\nu_2 - f_{,u} = -f_{,v}. \tag{29}$$

Substituting (10) into (29):

$$f_{,v} + g_{,v} = f_{,u} + g_{,u}. \tag{30}$$

Table II. Summary of conditions along the OG edge

Condition	Fractional flows		Mobilities
I	$f_{,v} = 0$	$\Leftrightarrow$	$\lambda_{w,v} = 0$
II	$g_{,v} - f_{,u} > 0$	$\Leftrightarrow$	$\lambda_{g,v} > \lambda_{w,u} + \lambda_{T,v}\frac{\lambda_g}{\lambda_T}$
II at O			$\lambda_{g,v} > \lambda_{w,u}$
II at G			$\lambda_{w,u} + \lambda_{o,v} = 0$

In terms of mobilities, Condition I above reads:

$$\lambda_{o,v} = \lambda_{o,u}. \tag{31}$$

As for the other two edges, this condition is identically satisfied by all models, because the oil mobility is identically zero along the WG edge. Using (30), the eigenvalues along the WG edge are given by:

$$\nu_1 = g_{,v} + f_{,v}, \qquad \nu_2 = g_{,v} - g_{,u}. \tag{32}$$

For strict hyperbolicity along the WG edge ($\nu_1 < \nu_2$), excluding the G vertex:

$$H_{wg} := -g_{,u} - f_{,v} > 0, \tag{33}$$

or, equivalently:

$$\frac{\lambda_w}{\lambda_T}(\lambda_{g,v} - \lambda_{g,u}) + \frac{\lambda_g}{\lambda_T}(\lambda_{w,u} - \lambda_{w,v}) > -\lambda_{o,u}. \tag{34}$$

The equality at the G vertex imposes that:

$$\lambda_{w,u} - \lambda_{w,v} = -\lambda_{o,u}. \tag{35}$$

Using Condition I along all three edges (Eqs. (14), (23) and (31)), the conditions at the W and G vertices reduce to (20) and (28), respectively. In Table III we summarize the conditions at the WG edge.

Remark. The conditions expressed in Tables I–III are *necessary* conditions for the type of behavior we propose, which leads to strict hyperbolicity of the system of

Table III. Summary of conditions along the WG edge

Condition	Fractional flows		Mobilities
I	$f_{,v} + g_{,v} = f_{,u} + g_{,u}$	$\Leftrightarrow$	$\lambda_{o,v} = \lambda_{o,u}$
II	$-g_{,u} - f_{,v} > 0$	$\Leftrightarrow$	$\frac{\lambda_w}{\lambda_T}(\lambda_{g,v} - \lambda_{g,u})$ $+ \frac{\lambda_g}{\lambda_T}(\lambda_{w,u} - \lambda_{w,v}) > -\lambda_{o,u}$
II at W			$\lambda_{g,v} > -\lambda_{o,u}$
II at G			$\lambda_{w,u} + \lambda_{o,v} = 0$

equations everywhere in the saturation triangle (with the exception of the G vertex, which is an umbilic point). They are not *sufficient* conditions.

3.3. A SIMPLE MODEL

Our interest here reduces to presenting a simple model that satisfies the conditions above. It is common practice to assume that relative permeabilities of the most and least wetting fluids (usually water and gas) depend only on their own saturation, whereas the relative permeability of the intermediate wetting fluid (usually oil) depends on all saturations. Although we do *not* defend this assumption in general, here we show it is possible to obtain models which are strictly hyperbolic everywhere in the three-phase flow region. We take, for example:

$$\lambda_w = \left(\frac{1}{\mu_w}\right) u^2, \tag{36}$$

$$\lambda_g = \left(\frac{1}{\mu_g}\right) (\beta_g v + (1 - \beta_g)v^2), \quad \beta_g > 0, \tag{37}$$

$$\lambda_o = \left(\frac{1}{\mu_o}\right) (1 - u - v)(1 - u)(1 - v). \tag{38}$$

The most important feature of the model is the positive derivative of the gas relative permeability function as it approaches zero. For the particular function used here, oil isoperms are slightly convex (Figure 4). It is immediate to check that the relative mobilities (36)–(38) satisfy Condition I on all three edges. Whether Condition II is satisfied will depend, in general, on the values of the fluid viscosities and the endpoint-slope of the gas relative permeability (Juanes, 2003).

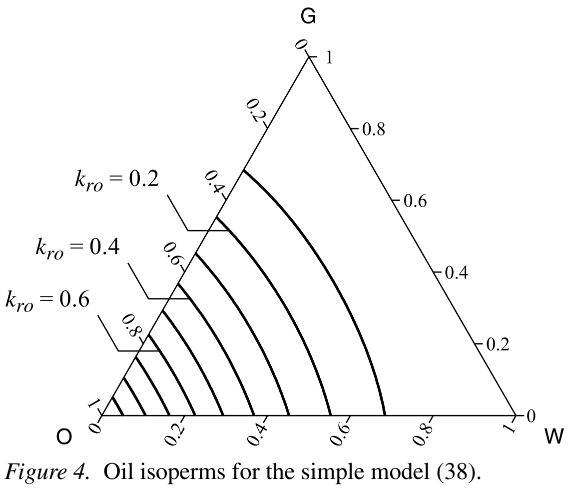


Figure 4. Oil isoperms for the simple model (38).

3.3.1. *Analysis Along the* OW *Edge*

The condition for strict hyperbolicity along the OW edge reads:

$$H_{ow} = g_{,v} - f_{,u} = \frac{1}{D(u)}\left[\frac{\beta_g}{\mu_g} - F(u)\right] > 0, \tag{39}$$

where

$$D(u) := \frac{1}{u^2/\mu_w + (1-u)^2/\mu_o}, \quad \text{and} \quad F(u) := 2\frac{u(1-u)}{\mu_o u^2 + \mu_w (1-u)^2}. \tag{40}$$

Defining $M := \max_{0<u<1} F(u)$, Condition II on OW is satisfied if:

$$\beta_g > \mu_g M. \tag{41}$$

Differentiating $F(u)$ with respect to u and equating to zero, and after some algebraic manipulations, one obtains that $M = 1/\bar{\mu}$, where $\bar{\mu} := \sqrt{\mu_o \mu_w}$ is the geometric mean of the water and oil viscosities. Therefore, the condition for strict hyperbolicity on the OW edge is:

$$\beta_g > \frac{\mu_g}{\bar{\mu}}, \qquad \bar{\mu} = \sqrt{\mu_o \mu_w}. \tag{42}$$

Despite the fact that Equation (42) is restricted to the simple model considered here (and therefore it is not valid in general), it is illuminating with regard to the required behavior for the relative permeability of the most nonwetting phase. Equation (42) expresses that there is a lower bound on the endpoint-slope of the nonwetting phase relative permeability, if the three-phase flow model is to be strictly hyperbolic. This threshold is proportional to the ratio between the viscosity of the nonwetting phase and the average viscosity of the other two phases. It is important to note the following *remarks*:

1. Condition (42) provides a *lower bound* of the endpoint slope for the system to be strictly hyperbolic. It is *not* an estimate of the actual value of this slope.
2. The dependence of the relative permeabilities on the viscosity ratio has been justified both theoretically and experimentally. We refer to Figure 7 in Odeh (1959) for experimental data that shows precisely the trend suggested here.

3.3.2. *Analysis Along the* OG *Edge*

For strict hyperbolicity along the OG edge (excluding the G vertex), the relevant condition is:

$$\begin{aligned} H_{og} \;=\; g_{,v} - f_{,u} &= \frac{1}{\mu_g \mu_o} \frac{1-v}{[(\beta_g v + (1-\beta_g)v^2)/\mu_g + (1-v)^2/\mu_o]^2} \cdot \\ &\quad \cdot (\beta_g + (2-\beta_g)v) > 0, \end{aligned} \tag{43}$$

which is always satisfied for all $v \in [0, 1)$, as long as the endpoint slope $\beta_g > 0$. At the G vertex ($u = 0,\ v = 1$), we obtain $H_{og} = 0$, so this point is an umbilic point, as required.

3.3.3. *Analysis Along the* WG *Edge*

The condition for strict hyperbolicity along this edge (not including the G vertex) is:

$$H_{wg} = -g_{,u} - f_{,v} = \frac{1}{E(v)}\left[\frac{C_{wg}(v)}{\mu_w\mu_g} + \frac{C_{og}(v)}{\mu_o\mu_g} + \frac{C_{wo}(v)}{\mu_w\mu_o}\right] > 0, \tag{44}$$

where,

$$E(v) = \left(\frac{\beta_g v + (1-\beta_g)v^2}{\mu_g} + \frac{(1-v)^2}{\mu_w}\right)^2, \tag{45}$$

$$C_{wg}(v) = (2 - (2-\beta_g)(1-v))(1-v), \tag{46}$$

$$C_{og}(v) = -(1 - (1-\beta_g)(1-v))v^2(1-v), \tag{47}$$

$$C_{ow}(v) = -v(1-v)^3. \tag{48}$$

At the G vertex, corresponding to $v = 1$, it is clear that $H_{wg} = 0$, as required. On the other hand, it is not easy to infer the conditions on the fluid viscosities and the endpoint-slope β_g such that the strict inequality (44) is satisfied on the entire edge. It is possible, however, to identify the conditions for strict hyperbolicity along this edge near the G vertex. Let $u = \varepsilon,\ v = 1 - \varepsilon$ with $\varepsilon \to 0$, that is, a state on the WG edge near the G corner. The first-order Taylor expansion of H_{wg} about $\varepsilon = 0$ is

$$H_{wg} = \mu_g\left[\frac{2}{\mu_w} - \frac{1}{\mu_o}\right]\varepsilon + O(\varepsilon^2). \tag{49}$$

Therefore, for $H_{wg} > 0$ in the neighborhood of the G corner, we obtain the condition

$$\mu_w < 2\mu_o. \tag{50}$$

This imposes an additional restriction (not obvious to anticipate) on the values of the fluid viscosities, if one wants the relative permeability model (36)–(38) to yield a strictly hyperbolic system. We emphasize that such condition was derived for this simple example only. Obviously, it should be understood as a limitation of the model, and not a physical restriction on the fluid viscosities.

4. Validation with Experimental Data

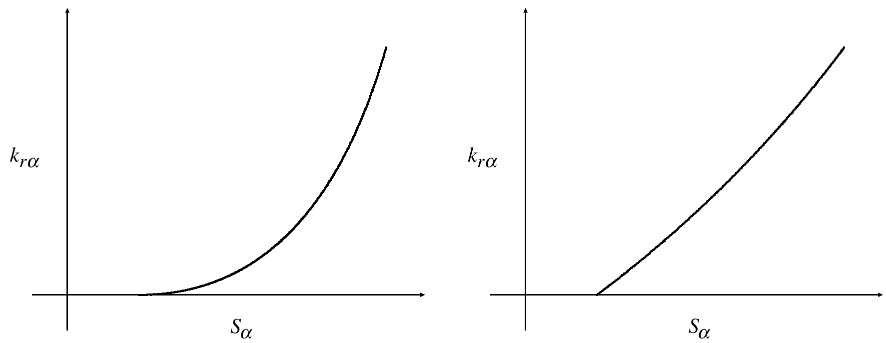
In Section 3 we derived the necessary conditions that must be satisfied by the relative permeabilities, if the system of equations describing three-phase flow is to be strictly hyperbolic everywhere inside the saturation triangle. The essential requirement for strict hyperbolicity turns out to be that the relative permeability of gas (the most nonwetting phase) must have a *positive* derivative with respect to its own saturation, at the edge of zero reduced gas saturation (Eq. (18)). In this section we verify how realistic this condition is, by means of comparison with experimental data. To this end, we use Oak's steady-state experiments (Oak, 1990; Oak *et al.*, 1990), which are arguably the most reliable and best-known data set available. The fact that we use steady-state relative permeability data in a dynamic fluid displacement model ought to be of little consequence because the relative permeabilities measured with many different methods are similar (Osoba *et al.*, 1951; Johnson *et al.*, 1959).

The data set consists of over 1800 two-phase and three-phase relative permeability measurements, obtained using a fully automated steady-state method. Three fired Berea sandstone cores were employed, with absolute permeabilities of 200 md (Sample 6), 800 md (Sample 14), and 1000 md (Sample 13). Water, oil, and gas viscosities were 1.06, 1.77, and 0.0187 cp, respectively. The study includes over 30 combinations of rock and fluid systems and saturation histories. A complete description of the experimental apparatus and procedure is given in the original references (Oak, 1990; Oak *et al.*, 1990).

4.1. DESCRIPTION OF THE 'ENDPOINT-SLOPE' ANALYSIS

We are interested in the qualitative behavior of the relative permeability of each phase in the region of low reduced saturation of that phase. More precisely, we want to determine whether the relative permeability of a phase, when expressed as a function of its own saturation only, takes off with a zero or a positive slope (see Figure 5).

Let us describe our 'endpoint-slope' analysis of Oak's relative permeability data. For each experiment (which consists of several – sometimes dozens –



(a) Zero-slope behavior (b) Positive-slope behavior

Figure 5. Markedly different qualitative behavior of the slope of the relative permeability of a phase, in the region near the 'immobile' saturation of that phase.

relative permeability measurements), we tabulate the relative permeability of a phase against its own saturation. For each phase α, we identify a maximum saturation $S_{\alpha,\max}$ that defines the range of saturations to be used in the analysis, and use only the data points for saturations $S_\alpha < S_{\alpha,\max}$. This range should be small enough to be considered close to the immobile saturation, but should have a sufficient number of data points to indicate a trend, given the precision of the measurements. We fit the power-law expression:

$$k_{r\alpha} = C_\alpha (S_\alpha - S_{\alpha i})^{m_\alpha}, \quad S_\alpha < S_{\alpha,\max}, \tag{51}$$

using a least squares procedure, with the following constraints:

$$C_\alpha > 0, \qquad S_{\alpha i} \geqslant 0, \qquad m_\alpha \geqslant 1. \tag{52}$$

The most relevant parameter is the exponent m_α. A value of one or close to one is indicative of a linear behavior of the relative permeability and, thus, a positive slope at the endpoint saturation. On the other hand, an exponent larger than two suggests that the relative permeability will approach a zero value (at the endpoint saturation) with zero slope. Below, we present the results of the analysis of four *representative* experiments.

4.2. TWO-PHASE FLOW EXPERIMENTS

4.2.1. *Primary Drainage Experiment*

The first experiment (Sample 13, Experiment 16a) corresponds to a *drainage* process, where gas is injected into an initially water-filled core, through a sequence of steady states. The relative permeability curves for this experiment are plotted in Figure 6 in semi-log scale. The results of the power-law fit of the data are presented in Table IV. The most important observation is the essential difference in the value of the exponent m_α for the wetting phase ($m_w \approx 8$) and the nonwetting phase

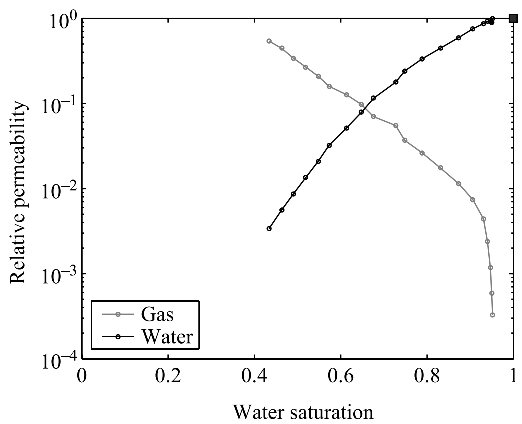
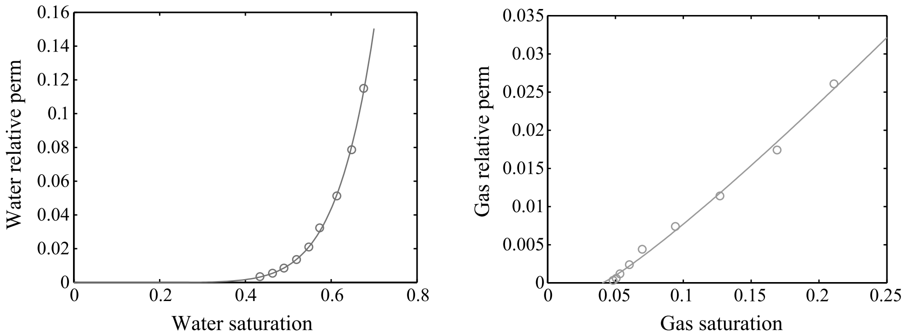


Figure 6. Relative permeability curves of water and gas for the two-phase drainage experiment (Sample 13, Experiment 16a of Oak's dataset). The solid square mark (■) indicates the initial saturation state of the core.

Table IV. Parameters of the power-law fit for the two-phase drainage experiment

	$S_{\alpha,\max}$	C_α	$S_{\alpha i}$	m_α
Water	0.7000	2.6362	0.0000	**8.0282**
Gas	0.2500	0.1902	0.0404	**1.1377**



(a) Water relative permeability (b) Gas relative permeability

Figure 7. Relative permeabilities as a function of their own saturation, for the two-phase drainage experiment.

($m_g \approx 1.1$). The actual fit is shown graphically in Figure 7. It is apparent that the water relative permeability reaches a value of zero with a zero value of the slope, whereas the gas relative permeability curve displays an almost linear relation against gas saturation, which can be assimilated to a positive slope near the critical gas saturation.

4.2.2. *Secondary Imbibition Experiment*

The data analyzed here (Sample 13, Experiment 16b) correspond to the *imbibition* process following the primary drainage experiment described in the previous paragraph. The relative permeability curves obtained in this way are plotted in semi-log scale in Figure 8. The results of the power-law fit to the data are presented in Table V. Once again, the values of the exponent m_α for the wetting phase ($m_w \approx 4$) and the nonwetting phase ($m_g = 1$) are fundamentally different. The actual fit to the relative permeability data is shown in Figure 9.

4.3. THREE-PHASE FLOW EXPERIMENTS

4.3.1. *Drainage-Dominated Experiment*

The first of the three-phase flow experiments (Sample 6, Experiment 15a) consists in a sequence of steady states of *increasing average gas saturation*, in which the ratio of gas/water flow rates is increased, while the water/oil ratio is held constant. The resulting saturation path for this experiment is shown on a ternary diagram in Figure 10. In Table VI we present the parameters of the power-law fit for the water

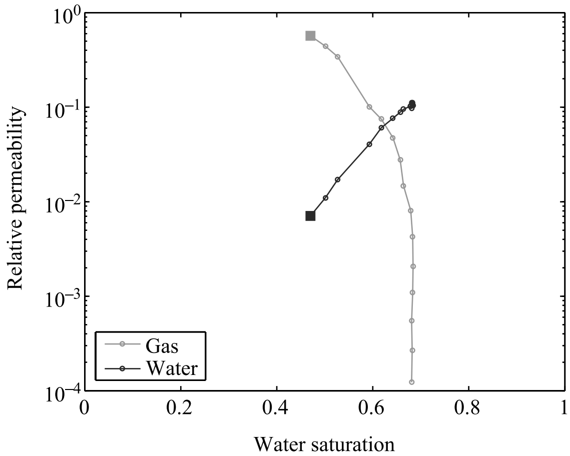
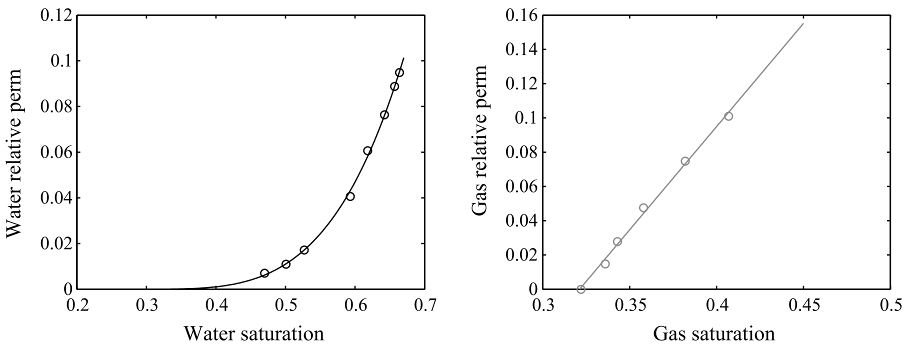


Figure 8. Relative permeability curves of water and gas for the two-phase imbibition experiment (Sample 13, Experiment 16b of Oak's dataset). The solid square mark (■) indicates the saturation state of the core after primary drainage.

Table V. Parameters of the power-law fit for the two-phase imbibition experiment

	$S_{\alpha,\max}$	C_α	$S_{\alpha i}$	m_α
Water	0.6700	3.9379	0.2785	**3.9030**
Gas	0.4500	1.2051	0.3212	**1.0000**



(a) Water relative permeability (b) Gas relative permeability

Figure 9. Relative permeabilities as a function of their own saturation, for the two-phase imbibition experiment.

and gas phases. As for the two-phase flow results, we observe a high value of the exponent ($m_w \approx 3$) for water, and a value close to one ($m_g \approx 1.2$) for gas. The actual fit is shown in Figure 11.

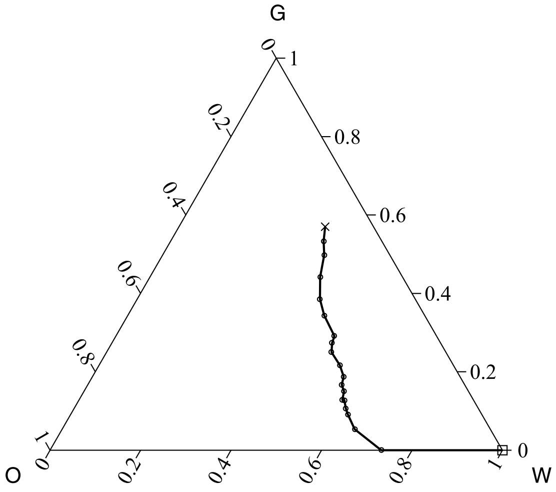
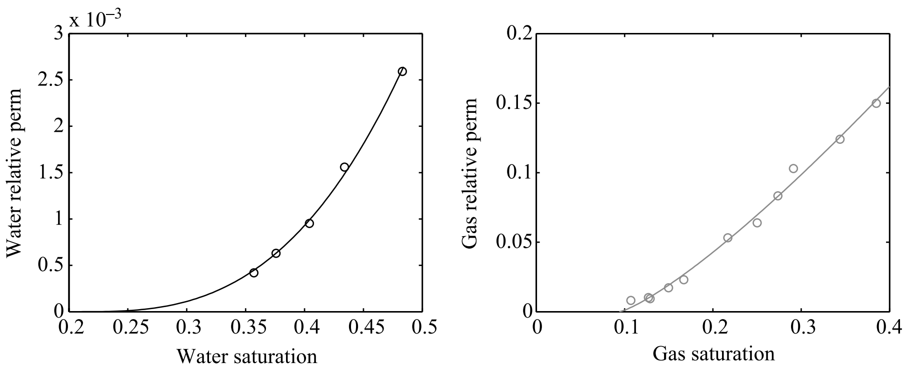


Figure 10. Saturation path for the drainage-dominated three-phase relative permeability experiment (Sample 6, Experiment 15a of Oak's dataset). The square mark (□) indicates the saturation state of the core at the beginning of the experiment.

Table VI. Parameters of the power-law fit for the three-phase drainage-dominated experiment

	$S_{\alpha,\max}$	C_α	$S_{\alpha i}$	m_α
Water	0.4840	0.1065	0.2092	**2.8623**
Gas	0.4000	0.7162	0.0941	**1.2552**



(a) Wate relative permeability (b) Gas relative permeability

Figure 11. Relative permeabilities as a function of their own saturation, for the three-phase drainage-dominated experiment.

4.3.2. *Imbibition-Dominated Experiment*

Our last example (Sample 6, Experiment 15b) is a three-phase flow experiment, consisting in a sequence of steady states of *increasing water saturation*. This

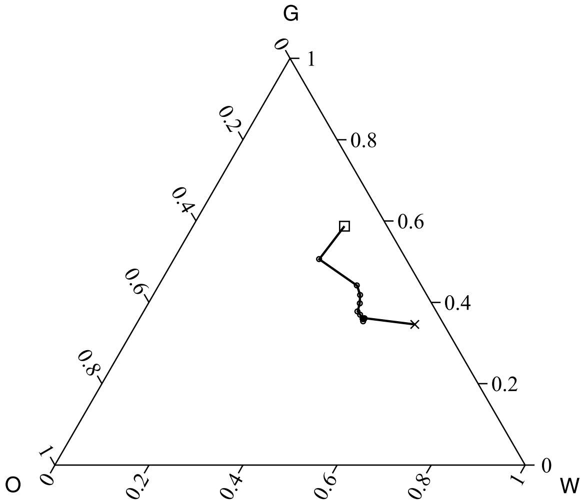
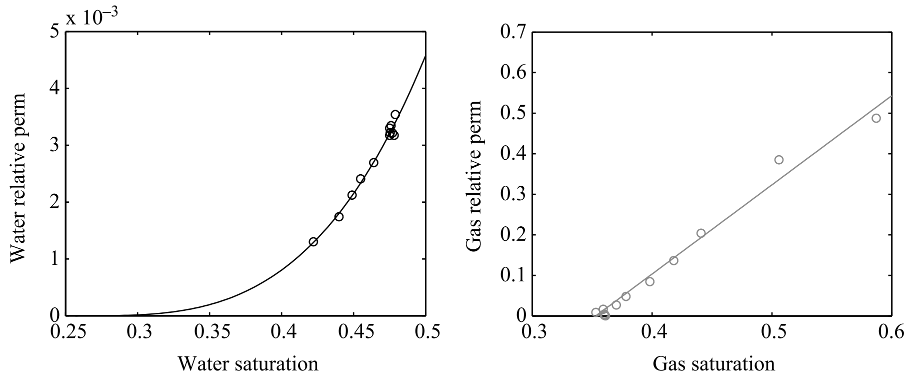


Figure 12. Saturation path for the imbibition-dominated three-phase relative permeability experiment (Sample 6, Experiment 15b of Oak's dataset). The square mark (□) indicates the saturation state of the core at the end of the drainage-dominated experiment.

Table VII. Parameters of the power-law fit for the three-phase imbibition-dominated experiment

	$S_{\alpha,\max}$	C_α	$S_{\alpha i}$	m_α
Water	0.5000	0.4737	0.2569	**3.2804**
Gas	0.6000	2.1985	0.3530	**1.0000**



(a) Water relative permeability (b) Gas relative permeability

Figure 13. Relative permeabilities as a function of their own saturation, for the three-phase imbibition-dominated experiment.

sequence is achieved by decreasing the ratio of gas/water flow rates, while keeping the water/oil ratio constant. In Figure 12 we depict the saturation path for this experiment. TableVII has the numeric values of the power-law fit, and Figure 13 shows the experimental data and the fitted relative permeability curves. The same qualitative behavior as that of the previous examples is observed ($m_w \approx 3$, $m_g = 1$).

Remark. The particular examples presented here are representative of more than one hundred experiments in Oak's dataset. His experimental data seem to corroborate the fundamental requirement for strict hyperbolicity of the model, that is, a positive slope of the relative permeability of the most nonwetting phase near its immobile saturation.

5. Conclusions

Traditional formulations of three-phase flow in porous media employ the usual extension of Darcy's equation. Within this framework, it was believed that elliptic regions were unavoidable when generic relative permeability functions were used in models of one-dimensional immiscible incompressible three-phase flow. This conclusion was inferred after a particular behavior of the relative permeabilities along the edges of the saturation triangle was assumed. In this paper we show it is possible to identify conditions on the relative permeabilities so that the system of equations is strictly hyperbolic everywhere in the saturation triangle. By means of a specific example, we suggest how strict hyperbolicity may be invoked to impose constraints on the parameters of the relative permeability model. It turns out that the fundamental requirement is a finite positive slope of the gas relative permeability at the saturation where gas becomes mobile. This condition is consistent with a pore-scale description of multiphase flow and, as shown in this paper, is also supported by experimental relative permeability data.

This important result is restricted to the case when gravitational effects are not accounted for, which is sensible only when the gravity number is small, or when flow is horizontal. It is possible, however, to extend this analysis to the case when flow is not horizontal and gravity is included, by allowing that relative permeabilities may vary (albeit slightly) with the gravity number (Juanes and Patzek, 2003).

We admit that we still do not have a definitive argument in favor or against the presence of elliptic regions in the saturation space, although we find good reasons to believe they are nothing else than mathematical artifacts of an incomplete mathematical model. In this paper, we review some of the facts suggesting that elliptic regions are unphysical in the context of three-phase flow displacements. Our current view is that one should use these hints to develop new relative permeability models, or an improved description altogether of three-phase flow in porous media.

Acknowledgements

We are grateful to Dr Dmitriy Silin for numerous insightful suggestions and his careful review of the manuscript. We also thank Prof Martin Blunt for his comments and the vigorous discussion on elliptic regions, and for providing an electronic version of Oak's relative permeability data. This work was supported by the Laboratory Directed Research and Development Program of Lawrence Berkeley National Laboratory under the Department of Energy Contract No. DE-AC03-76SF00098. Funding provided by Barrié de la Maza, Jane Lewis, and Repsol-YPF fellowships, awarded to the first author, is also gratefully acknowledged.

References

Avraam, D. G. and Payatakes, A. C.: 1995, Flow regimes and relative permeabilities during steady-state 2-phase flow in porous-media, *J. Fluid Mech.* **293**, 207–236.

Azevedo, A. V. and Marchesin, D.: 1995, Multiple viscous solutions for systems of conservation laws, *Trans. Am. Math. Soc.* **347**(8), 3061–3077.

Azevedo, A. V., Marchesin, D., Plohr, B. and Zumbrun, K.: 2002, Capillary instability in models for three-phase flow, *Z. Angew. Math. Phys.* **53**, 713–746.

Barenblatt, G. I., Entov, V. M. and Ryzhik, V. M.: 1990, *Theory of Fluid Flows through Natural Rocks*, Kluwer, Dordrecht.

Bell, J. B., Trangenstein, J. A. and Shubin, G. R.: 1986, Conservation laws of mixed type describing three-phase flow in porous media, *SIAM J. Appl. Math.* **46**(6), 1000–1017.

Charny, I. A.: 1963, *Subterranean Hydro-Gas Dynamics*, Gostoptekhizdat, Moscow (in Russian).

Coddington, E. A. and Levinson, N.: 1955, *Theory of Ordinary Differential Equations*, McGraw Hill, New York.

Falls, A. H. and Schulte, W. M.: 1992, Theory of three component, three phase displacement in porous media, *SPE Reserv. Eng.* **7**(3), 377–384.

Fayers, F. J.: 1987, Extension of Stone's method I and conditions for real characteristics in three-phase flow, in: *SPE Annual Technical Conference and Exhibition*, Dallas, TX (SPE 16965).

Fayers, F. J. and Matthews, J. D.: 1984, Evaluation of normalized Stone's methods for estimating three-phase relative permeabilities, *Soc. Pet. Eng. J.* **24**(2), 224–232; *Petrol. Trans. AIME*, 277.

Geffens, T. M., Owens, W. W., Parrish, D. R. and Morse, R. A.: 1951, Experimental investigation of factors affecting laboratory relative permeability measurements, *Petrol. Trans. AIME*, **192**, 99–110.

Grader, A. S. and O'Meara Jr., D. J.: 1988, Dynamic displacement measurements of three-phase relative permeabilities using three immiscible fluids, in: *SPE Annual Technical Conference and Exhibition*, Houston, TX (SPE 18293).

Gray, W. G. and Hassanizadeh, S. M.: 1998, Macroscale continuum mechanics for multiphase porous-media flow including phases, interfaces, common lines and common points, *Adv. Water Resour.* **21**, 261–281.

Guzmán, R. E. and Fayers, F. J.: 1997, Mathematical properties of three-phase flow equations, *Soc. Pet. Eng. J.* **2**(3), 291–300.

Hassanizadeh, S. M.: 1986, Derivation of basic equations of mass transport in porous media, Part 2. Generalized Darcy's and Fick's laws, *Adv. Water Resour.* **9**, 207–222.

Hassanizadeh, S. M. and Gray, W. G.: 1993, Toward an improved description of the physics of two-phase flow, *Adv. Water Resour.* **16**(1), 53–67.

Hicks Jr., P. J. and Grader, A. S.: 1996, Simulation of three-phase displacement experiments, *Transp. Porous Media* **24**, 221–245.

Holden, L.: 1990, On the strict hyperbolicity of the Buckley–Leverett equations for three-phase flow in a porous medium, *SIAM J. Appl. Math.* **50**(3), 667–682.

Holden, H., Holden, L. and Risebro, N. H.: 1990, Some qualitative properties of 2×2 systems of conservation laws of mixed type, in: B. L. Keyfitz and M. Shearer (eds), *Nonlinear Evolution Equations that Change Type*, Springer-Verlag, New York, pp. 67–78.

Jackson, M. D. and Blunt, M. J.: 2002, Elliptic regions and stable solutions for three-phase flow in porous media, *Transp. Porous Media* **48**, 249–269.

Johnson, E. F., Bossler, D. P. and Naumann, V. O.: 1959, Calculation of relative permeability from displacement experiments, *Petrol. Trans. AIME* **216**, 370–372.

Juanes, R.: 2003, Displacement theory and multiscale numerical modeling of three-phase flow in porous media, PhD Dissertation, University of California at Berkeley.

Juanes, R. and Patzek, T. W.: 2003, Relative permeabilities in co-current three-phase displacements with gravity, in: *SPE Western Regional/AAPG Pacific Section Joint Meeting*, Long Beach, CA (SPE 83445).

Juanes, R. and Patzek, T. W.: 2004, Analytical solution to the Riemann problem of three-phase flow in porous media, *Transp. Porous Media* **55**(1), 47–70.

Keyfitz, B. L.: 1990, Shocks near the sonic line: a comparison between steady and unsteady models for change of type, in: B. L. Keyfitz and M. Shearer (eds), *Nonlinear Evolution Equations that Change Type*, Springer-Verlag, New York, pp. 89–106.

Lenhard, R. J. and Parker, J. C.: 1987, A model for hysteretic constitutive relations governing multiphase flow, 2. Permeability-saturation relations, *Water Resour. Res.* **23**(2), 2197–2206.

Lenormand, R., Touboul, E. and Zarcone, C.: 1988, Numerical models and experiments on immiscible displacements in porous media, *J. Fluid Mech.* **189**, 165–187.

Leverett, M. C. and Lewis, W. B.: 1941, Steady flow of gas-oil-water mixtures through unconsolidated sands, *Petrol. Trans. AIME* **142**, 107–116.

Marchesin, D. and Medeiros, H. B.: 1989, A note on the stability of eigenvalue degeneracy in nonlinear conservation laws of multiphase flow, in: W. B. Lindquist (ed.), *Current Progress in Hyperbolic Systems: Riemann Problems and Computations*, American Mathematical Society, Providence, RI, pp. 215–224.

Marchesin, D. and Plohr, B. J.: 2001, Wave structure in WAG recovery, *Soc. Pet. Eng. J.* **6**(2), 209–219.

Muskat, M.: 1949, *Physical Principles of Oil Production*, McGraw-Hill, New York.

Oak, M. J.: 1990, Three-phase relative permeability of water-wet Berea, in: *SPE/DOE Seventh Symposium on Enhanced Oil Recovery*, Tulsa, OK (SPE/DOE 20183).

Oak, M. J., Baker, L. E. and Thomas, D. C.: 1990, Three-phase relative permeability of Berea sandstone, *J. Pet. Technol.* **42**(8), 1054–1061.

Odeh, A. S.: 1959, Effect of viscosity ratio on relative permeability, *Petrol. Trans. AIME* **216**, 346–353.

Osoba, J. S., Richardson, J. G., Kerver, J. K., Hafford, J. A. and Blair, P. M.: 1951, Laboratory measurements of relative permeability, *Petrol. Trans. AIME* **192**, 47–56.

Sahni, A., Guzmán, R. and Blunt, M.: 1996, Theoretical analysis of three phase flow experiments in porous media, in: *SPE Annual Technical Conference and Exhibition*, Denver, CO (SPE 36664).

Shearer, M.: 1988, Loss of strict hyperbolicity of the Buckley-Leverett equations for three phase flow in a porous medium, in: M. F. Wheeler (ed.), *Numerical Simulation in Oil Recovery*, Springer-Verlag, New York, pp. 263–283.

Shearer, M. and Trangenstein, J. A.: 1989, Loss of real characteristics for models of three-phase flow in a porous medium, *Transp. Porous Media* **4**, 499–525.

Silin, D. and Patzek, T.: 2004, On Barenblatt's model of spontaneous countercurrent imbibition, *Transp. Porous Media* **54**(3), 297–322.

Stklyanin, Y. I.: 1960, The motion of a mixture of three liquids in a porous medium (in Russian), *Izv. Akad. Nauk SSSR, Otd. Tekh. Nauk, Mekhanika i Mashinostroenie* **2**(5).

Stone, H. L.: 1970, Probability model for estimating three-phase relative permeability, *J. Pet. Technol.* **23**(2), 214–218; *Petrol. Trans. AIME* 249.

Trangenstein, J. A.: 1989, Three-phase flow with gravity, in: W. B. Lindquist (ed.), *Current Progress in Hyperbolic Systems: Riemann Problems and Computations*, American Mathematical Society, Providence, RI, pp. 147–159.

Truesdell, C. and Noll, W.: 1965, *The Non-Linear Field Theories of Mechanics* (Second edition, 1992), Springer-Verlag, Berlin.

Wyckoff, R. D. and Botset, H. G.: 1936, The flow of gas-liquid mixtures through unconsolidated sands, *Physics* **7**, 325–345.

Yortsos, Y. C., Xu, B. and Salin, D.: 1997, Phase diagram of fully-developed drainage in porous media, *Phys. Rev. Lett.* **79**(23), 4581–4584.

Zauderer, E.: 1983, *Partial Differential Equations of Applied Mathematics*, John Wiley & Sons, New York.